Theorem 1. *[Exercise 1.18] Let M be a topological manifold. Then any two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

Proof. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two smooth atlases for M that determine the same smooth structure $\mathcal{A}$. Then $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$, so $\mathcal{A}_1 \cup \mathcal{A}_2$ must be a smooth atlas since every chart in $\mathcal{A}_1$ is compatible with every chart in $\mathcal{A}_2$. Conversely, if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas then the smooth structures determined by $\mathcal{A}_1$ and $\mathcal{A}_2$ both contain $\mathcal{A}_1 \cup \mathcal{A}_2$. But there is exactly one smooth structure containing $\mathcal{A}_1 \cup \mathcal{A}_2$, so $\mathcal{A}_1$ and $\mathcal{A}_2$ determine the same smooth structure. ☐

Theorem 2. *[Exercise 1.44] Let M be a smooth n-manifold with boundary and let U be an open subset of M.*

(1) *U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, φ) for M such that $V \subseteq U$ defines a smooth structure on U. With this topology and smooth structure, U is called an* ***open submanifold with boundary****.*
(2) *If $U \subseteq \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an* ***open submanifold of*** *M.*
(3) $\operatorname{Int} M$ *is an open submanifold of M (without boundary).*

Proof. Parts (1) and (2) are obvious. Part (3) follows from (2) and the fact that $\operatorname{Int} M$ is an open subset of M. ☐

Theorem 3. *[Problem 1-6] Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, then it has uncountably many distinct ones.*

Proof. We use the fact that for any $s > 0$, $F_s(x) = |x|^{s-1} x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$. Let $\mathcal{A}$ be a smooth structure for M and choose some coordinate map $\varphi : U \to \mathbb{B}^n$ centered at some $p \in U$. Let $\mathcal{A}'$ be the smooth atlas obtained by replacing every coordinate map $\psi : V \to \mathbb{R}$ in $\mathcal{A}$ with $\psi' : V \cap (M \setminus \{p\}) \to \mathbb{R}$, except when $\psi = \varphi$. For any $s > 0$, let $\mathcal{A}_s$ be the smooth atlas obtained from $\mathcal{A}'$ by replacing φ with $F_s \circ \varphi$. Since F_s is a diffeomorphism if and only if $s = 1$, the smooth structures determined by $\mathcal{A}_s$ and $\mathcal{A}_t$ are the same if and only if $s = t$. This shows that there are uncountably many distinct smooth structures on M. ☐

Theorem 4. *[Problem 1-8] An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. There exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.*

Proof. The first part follows from the lifting criterion (Proposition A.78). By rotating $\mathbb{R}^2$ appropriately, we may assume that $N = (0,1) \notin U$. Let $\sigma : \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}$ be the stereographic projection given by $\sigma(x^1, x^2) = x^1/(1-x^2)$. We can compute

$$(\sigma \circ \theta^{-1})(\alpha) = \frac{\cos\alpha}{1-\sin\alpha},$$

which is a diffeomorphism on $\theta(U)$. □

Theorem 5. *[Problem 1-10] Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\begin{pmatrix} I_k \\ B \end{pmatrix}$, where I_k denotes the $k \times k$ identity matrix.*

Proof. Let $\mathcal{B} = \{e_1, \ldots, e_k\}$. The matrix of $\varphi(S)$ represents the linear map $(\pi_Q|_S) \circ (\pi_P|_S)^{-1}$. Since $\pi_P|_S$ is an isomorphism, the vectors $(\pi_P|_S)^{-1}(\mathcal{B})$ form a basis for S. But then

$$\begin{aligned}(\mathrm{Id}_{\mathbb{R}^n}|_S \circ (\pi_P|_S)^{-1})(\mathcal{B}) &= ((\pi_P|_S + \pi_Q|_S) \circ (\pi_P|_S)^{-1})(\mathcal{B}) \\ &= (\mathrm{Id}_P + \pi_Q|_S \circ (\pi_P|_S)^{-1})(\mathcal{B})\end{aligned}$$

is a basis for S. This is precisely the result desired. □

Theorem 6. *[Problem 1-12] Suppose $M_1, \ldots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.*

Proof. Denote the dimensions of $M_1, \ldots, M_k, N$ by $n_1, \ldots, n_k, d$. Let $p = (p_1, \ldots, p_k, q) \in M_1 \times \cdots \times M_k \times N$, let $\varphi_i : M_i \to \mathbb{R}^{n_i}$ be coordinate maps around p_i for $i = 1, \ldots, k$, and let $\psi : N \to \mathbb{H}^d$ be a coordinate map around q. Then $\varphi_1 \times \cdots \times \varphi_k \times \psi : M_1 \times \cdots \times M_k \times N \to \mathbb{H}^{n_1 + \cdots + n_k + d}$ is a coordinate map, and $(\varphi_1 \times \cdots \times \varphi_k \times \psi)(p_1, \ldots, p_k, q) \in \partial\mathbb{H}^{n_1 + \cdots + n_k + d}$ if and only if $\psi(q) \in \partial\mathbb{H}^d$, which is true if and only if $q \in \partial N$. □

Chapter 2. Smooth Maps

Theorem 7. *[Exercise 2.1] Let M be a smooth manifold with or without boundary. Then pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.*

Proof. The product of two smooth functions is also smooth. □

Theorem 8. *[Exercise 2.2] Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Then a function $f : U \to \mathbb{R}^k$ is smooth in the sense of smooth manifolds if and only if it is smooth in the sense of ordinary calculus.*

Proof. Obvious since the single chart $\mathrm{Id}_{\mathbb{R}^n}$ covers $\mathbb{R}^n$. $\square$

Theorem 9. *[Exercise 2.3] Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Then $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for* every *smooth chart (U, φ) for M.*

Proof. If $p \in U$ then there is a smooth chart (V, ψ) for M such that $f \circ \psi^{-1}$ is smooth and $p \in V$. But on $\varphi(U \cap V)$ we have

$$f \circ \varphi^{-1} = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1},$$

which is smooth since $\psi \circ \varphi^{-1}$ is smooth. This shows that $f \circ \varphi^{-1}$ is smooth in a neighborhood of every point in $\varphi(U)$, so $f \circ \varphi^{-1}$ is smooth. $\square$

Theorem 10. *[Exercise 2.7(1)] Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:*

1. *For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\left(U \cap F^{-1}(V)\right)$ to $\psi(V)$.*
2. *F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\left(U_\alpha \cap F^{-1}(V_\beta)\right)$ to $\psi_\beta(V_\beta)$.*

Proof. (1) is obvious, as is (2) $\Rightarrow$ (1). If (1) holds then F is continuous by Proposition 2.4, and we can choose the smooth atlases to be the smooth structures for M and N. For every α and β the map $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth since for all $p \in M$ we can find smooth coordinate maps φ containing p and ψ containing $F(p)$ such that

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} = \psi_\beta \circ \psi^{-1} \circ \psi \circ F \circ \varphi \circ \varphi^{-1} \circ \varphi_\alpha^{-1}$$

on a small neighborhood around $\varphi_\alpha(p)$; this map is smooth since the above charts are all smoothly compatible. $\square$

Theorem 11. *[Exercise 2.7(2)] Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.*

1. *If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.*
2. *Conversely, if F is smooth, then its restriction to every open subset is smooth.*

Proof. (1) is obvious. For (2), let E be an open subset of M. Let $p \in E$ and choose smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth. By restricting φ to $U \cap E$, we find that $F|_E$ is also smooth. □

Theorem 12. *[Exercise 2.9] Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary. Then the coordinate representation of F with respect to* every *pair of smooth charts for M and N is smooth.*

Proof. Almost identical to Theorem 9. □

Theorem 13. *[Exercise 2.11] Let M, N and P be smooth manifolds with or without boundary.*

1. *Every constant map $c : M \to N$ is smooth.*
2. *The identity map of M is smooth.*
3. *If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.*

Proof. If $M \neq \emptyset$ then let y be the constant value that c assumes and let (V, ψ) be a smooth chart containing y. If $x \in M$ and (U, φ) is any smooth chart containing x then $\psi \circ c \circ \varphi^{-1}$ is constant and therefore smooth. This proves (1). Part (2) follows from the fact that any two coordinate maps for M are smoothly compatible. For (3), if $p \in U$ and (E, φ) is any smooth chart in U containing p then $\varphi \circ \varphi^{-1} = \mathrm{Id}_{\varphi(E)}$ is smooth. □

Theorem 14. *[Exercise 2.16]*

1. *Every composition of diffeomorphisms is a diffeomorphism.*
2. *Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.*
3. *Every diffeomorphism is a homeomorphism and an open map.*
4. *The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.*
5. *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.*

Proof. If $f : M \to N$ and $g : N \to P$ are diffeomorphisms then $f^{-1} \circ g^{-1}$ is a smooth inverse to $g \circ f$, which proves (1). If $f_i : M_i \to N_i$ are diffeomorphisms for $i = 1, \ldots, n$ then $f_1^{-1} \times \cdots \times f_n^{-1}$ is a smooth inverse to $f_1 \times \cdots \times f_n$, which proves (2). Part (3) follows from the fact that smooth maps are continuous, and part (4) follows immediately from (3) and Theorem 11. Part (5) follows from (1). □

Theorem 15. *[Exercise 2.19] Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\operatorname{Int} M$ to $\operatorname{Int} N$.*

Proof. It suffices to show that $F(\partial M) \subseteq \partial N$, for then $F(\operatorname{Int} M) = \operatorname{Int} N$ by Theorem 1.46 and $F|_{\operatorname{Int} M}$ is a diffeomorphism by Theorem 14. Let $x \in \partial M$ and choose smooth charts (U, φ) containing x and (V, ψ) containing $F(x)$ such that $\varphi(x) \in \partial \mathbb{H}^n$, $F(U) = V$ and $\psi \circ F \circ \varphi^{-1}$ is a diffeomorphism. Then $\varphi \circ F^{-1}$ is a smooth coordinate map sending $F(x)$ to $\varphi(x) \in \partial \mathbb{H}^n$, so $F(x) \in \partial N$ by definition. □

Example 16. [Exercise 2.27] Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.

Take $f(x) = 1/x$ defined on $\mathbb{R} \setminus \{0\}$.

Example 17. [Problem 2-1] Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\left(U \cap f^{-1}(V)\right)$ to $\psi(V)$, but f is not smooth as defined in this chapter.

f is clearly smooth on $\mathbb{R} \setminus \{0\}$. If $x = 0$ then $\varphi = \operatorname{Id}_{(-1,1)}$ is a coordinate map containing x and $\psi = \operatorname{Id}_{(1/2,3/2)}$ is a coordinate map containing $f(x) = 1$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth on $\varphi\left((-1,1) \cap f^{-1}\left((1/2, 3/2)\right)\right) = \varphi\left([0,1)\right) = [0,1)$. However, f is clearly not smooth since it is not continuous at 0.

Theorem 18. *Let $M_1, \ldots, M_{k-1}$ be smooth manifolds without boundary and let M_k be a smooth manifold with boundary. Then each projection map $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth.*

Proof. Let n_i be the dimension of M_i. Let $x = (x_1, \ldots, x_k) \in M_1 \times \cdots \times M_k$. For each $i = 1, \ldots, k$, choose a smooth chart (U_i, φ_i) containing x_i. Write $U = U_1 \times \cdots \times U_k$ and $\varphi = \varphi_1 \times \cdots \times \varphi_k$. Then (U, φ) is a smooth chart containing x and $\varphi_i \circ \pi_i \circ \varphi^{-1} = p_i$ is smooth, where p_i is a projection from $\mathbb{R}^{n_1} \times \cdots \times \mathbb{H}^{n_k}$ to $\mathbb{R}^{n_i}$ (or $\mathbb{H}^{n_i}$ if $i = k$). This shows that π_i is smooth. □

Theorem 19. *[Problem 2-2] Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.*

Proof. For simplicity we assume that M_k is a smooth manifold with boundary. From Proposition 2.10(d) it is clear that every F_i is smooth if F is smooth. Suppose that each F_i is smooth and let $x \in N$. For each i, choose smooth charts (U_i, φ_i) containing x and (V_i, ψ_i) containing $F_i(x)$ such that $\psi_i \circ F_i \circ \varphi_i^{-1} = \psi_i \circ \pi_i \circ F \circ \varphi_i^{-1}$. Replace U_1 with

$U = U_1 \cap \cdots \cap U_k$ and replace φ_1 with $\varphi_1|_U$; by Theorem 12, each map $\psi_i \circ \pi_i \circ F \circ \varphi_1^{-1}$ is smooth. Write $V = V_1 \times \cdots \times V_k$ and $\psi = \psi_1 \times \cdots \times \psi_k$ so that (V, ψ) is a smooth chart containing $F(x)$. Since the ith component of $\psi \circ F \circ \varphi_1^{-1}$ is just $\psi_i \circ \pi_i \circ F \circ \varphi_1^{-1}$, the map $\psi \circ F \circ \varphi_1^{-1}$ is smooth. This shows that F is smooth. □

Theorem 20. *[Problem 2-4] The inclusion map $f : \overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.*

Proof. It is clear that f is smooth on $\operatorname{Int} \overline{\mathbb{B}}^n = \mathbb{B}^n$. For points on $\partial\mathbb{B}^n$, the map provided by Problem 3-4 in [[1]] is a suitable coordinate map. □

Theorem 21. *[Problem 2-7] Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Then the vector space $C^\infty(M)$ is infinite-dimensional.*

Proof. We first show that any collection $\{f_\alpha\}$ from $C^\infty(M)$ with nonempty disjoint supports is linearly independent. Suppose there exist $r_1, \ldots, r_k \in \mathbb{R}$ such that

$$r_1 f_{\alpha_1} + \cdots + r_k f_{\alpha_k} = 0$$

where $\alpha_1, \ldots, \alpha_k$ are distinct. For each i, choose some $x_i \in \operatorname{supp} f_{\alpha_i}$ such that $f_{\alpha_i}(x_i) \neq 0$. Then

$$r_i f_{\alpha_i}(x_i) = (r_1 f_{\alpha_1} + \cdots + r_k f_{\alpha_k})(x_i) = 0,$$

so $r_i = 0$ since $f_{\alpha_i}(x_i) \neq 0$.

Choose some coordinate cube $U \subseteq M$ and a homeomorphism $\varphi : U \to (0,1)^n$. For each $i = 1, 2, \ldots,$ let $B_i = (0,1)^{n-1} \times (1/(i+1), 1/i)$ and let A_i be any nonempty closed subset of B_i. By Proposition 2.25, there is a smooth bump function f_i for $\varphi^{-1}(A_i)$ supported in $\varphi^{-1}(B_i)$. Furthermore, since the sets $\{\varphi^{-1}(B_i)\}$ are disjoint, the functions $\{f_i\}$ are linearly independent. This shows that $C^\infty(M)$ is infinite-dimensional. □

Theorem 22. *[Problem 2-8] Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1, \ldots, x^n) = [x^1, \ldots, x^n, 1]$. Then F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. A similar statement holds for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G(z^1, \ldots, z^n) = [z^1, \ldots, z^n, 1]$.*

Proof. The inverse to F is given by

$$F^{-1}[y^1, \ldots, y^n, y^{n+1}] = (y^1/y^{n+1}, \ldots, y^n/y^{n+1});$$

it is easy to check that the two maps are smooth since F^{-1} itself is a coordinate map for $\mathbb{RP}^n$. The image of F is dense in $\mathbb{RP}^n$ since any point $[x^1, \ldots, x^n, 0]$ not in the image of F can be approximated by some sequence $[x^1, \ldots, x^n, a_k]$ where $a_k \to 0$. □

Theorem 23. *[Problem 2-9] Given a polynomial p in one variable with complex coefficients, not identically zero, there is a unique smooth map $\widetilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map of Theorem 22:*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "p"'] & \mathbb{CP}^1 \arrow[d, "\widetilde{p}"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{CP}^1 \end{tikzcd}$$

Proof. We can assume that p is not constant, for otherwise the result is clear. For any $z = [z_1, z_2] \in \mathbb{CP}^1$, define

$$\widetilde{p}(z) = \begin{cases} [p(z_1/z_2), 1], & z_2 \neq 0, \\ [1, 0], & z_2 = 0. \end{cases}$$

The map $\widetilde{p}$ clearly satisfies the given diagram and is smooth on the image of G, so it remains to show that $\widetilde{p}$ is smooth in a neighborhood of $[1, 0]$. Define a diffeomorphism $H : \mathbb{C} \to \mathbb{CP}^1$ by

$$H(z) = [1, z];$$

by definition, $(H(\mathbb{C}), H^{-1})$ is a smooth chart for $\mathbb{CP}^1$. We can compute

$$\begin{aligned} (H^{-1} \circ \widetilde{p} \circ H)(z) &= (H^{-1} \circ \widetilde{p})[1, z] \\ &= \begin{cases} H^{-1}[p(1/z), 1], & z \neq 0, \\ H^{-1}[1, 0], & z = 0, \end{cases} \\ &= \begin{cases} 1/p(1/z), & z \neq 0, \\ 0, & z = 0, \end{cases} \end{aligned}$$

and it is easy to check that this map is smooth in a neighborhood of 0 whenever p is non-constant (since $1/p(1/z)$ is a rational function in z). This shows that $\widetilde{p}$ is smooth. □

Theorem 24. *[Problem 2-10] For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.*

(1) *F^* is a linear map.*
(2) *Suppose M and N are smooth manifolds. Then $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.*
(3) *Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Then it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.*

Proof. (1) is obvious. For (2), if F is smooth and $f \in C^\infty(N)$ then $F^*(f) = f \circ F \in C^\infty(M)$ by Proposition 2.10(d). Conversely, suppose that $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $x \in M$ and let n be the dimension of N. Choose smooth charts (U, φ) containing x and (V, ψ) containing $F(x)$ such that $F(U) \subseteq V$. For each $i = 1, \ldots, n$ the map $\pi_i \circ \psi \circ F = F^*(\pi_i \circ \psi)$ is smooth, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the ith coordinate. By 9, $\pi_i \circ \psi \circ F \circ \varphi^{-1}$ is smooth for $i = 1, \ldots, n$, so $\psi \circ F \circ \varphi^{-1}$ must be smooth. This shows that F is smooth. Part (3) follows directly from part (2): if F is a diffeomorphism then $(F^{-1})^* : C^\infty(M) \to C^\infty(N)$ is clearly an inverse to F^*, and conversely if $F^*|_{C^\infty(N)}$ is an isomorphism then F and F^{-1} are both smooth. □

Theorem 25. *[Problem 2-11] Suppose V is a real vector space of dimension $n \geq 1$. Define the* ***projectivization of*** *V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. Then $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \ldots, v^n] = [v^i E_i]$ is a diffeomorphism.*

Proof. Obvious. □

Theorem 26. *[Problem 2-14] Suppose A and B are disjoint closed subsets of a smooth manifold M. There exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.*

Proof. By Theorem 2.29, there are functions $f_A, f_B : M \to \mathbb{R}$ such that $f_A^{-1}(\{0\}) = A$ and $f_B^{-1}(\{0\}) = B$; the function defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

satisfies the required properties. □

Chapter 3. Tangent Vectors

Theorem 27. *[Exercise 3.7] Let M, N and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear.*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.*
3. *$d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.*
4. *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Proof. If $v, w \in T_pM$ and $c \in \mathbb{R}$ then

$$dF_p(v+w)(f) = (v+w)(f \circ F) = v(f \circ F) + w(f \circ F) = dF_p(v)(f) + dF_p(w)(f)$$

and

$$dF_p(cv)(f) = (cv)(f \circ F) = c dF_p(v)(f)$$

for all $f \in C^\infty(N)$, which proves (1). To prove (2), we have

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f)$$

for all $f \in C^\infty(N)$ and $v \in T_pM$, so $d(G \circ F)_p = dG_{F(p)} \circ dF_p$. For (3),

$$d(\mathrm{Id}_M)_p(v)(f) = v(f \circ \mathrm{Id}_M) = v(f),$$

so $d(\mathrm{Id}_M)_p(v) = v$ for all $v \in T_pM$ and therefore $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$. Part (4) follows immediately, since

$$d(F^{-1})_{F(p)} \circ dF_p = d(F^{-1} \circ F)_p = d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$$

and similarly $dF_p \circ d(F^{-1})_{F(p)} = \mathrm{Id}_{T_{F(p)}N}$. □

Example 28. [Exercise 3.17] Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\widetilde{x}, \widetilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\widetilde{x} = x, \qquad \widetilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \widetilde{x}}\right|_p,$$

even though the coordinate functions x and $\widetilde{x}$ are identically equal.

Since an inverse to $(x, y) \mapsto (x, y + x^3)$ is given by $(\widetilde{x}, \widetilde{y}) \mapsto (\widetilde{x}, \widetilde{y} - \widetilde{x}^3)$, the coordinates are smooth and global. Now

$$\begin{aligned} \left.\frac{\partial}{\partial x}\right|_p &= \left.\frac{\partial}{\partial \widetilde{x}}\right|_p + 3 \left.\frac{\partial}{\partial \widetilde{y}}\right|_p, \\ \left.\frac{\partial}{\partial y}\right|_p &= \left.\frac{\partial}{\partial \widetilde{y}}\right|_p, \end{aligned}$$

so $\partial/\partial x|_p \neq \partial/\partial \widetilde{x}|_p$.

Theorem 29. *Let X be a connected topological space and let $\sim$ be an equivalence relation on X. If every $x \in X$ has a neighborhood U such that $p \sim q$ for every $p, q \in U$, then $p \sim q$ for every $p, q \in X$.*

Proof. Let $p \in X$ and let $S = \{q \in X : p \sim q\}$. If $q \in S$ then there is a neighborhood U of q such that $q_1 \sim q_2$ for every $q_1, q_2 \in U$. In particular, for every $r \in U$ we have $p \sim q$ and $q \sim r$ which implies that $p \sim r$, and $U \subseteq S$. This shows that S is open. If $q \in X \setminus S$ then there is a neighborhood U of q such that $q_1 \sim q_2$ for every $q_1, q_2 \in U$. If $p \sim r$ for some $r \in U$ then $p \sim q$ since $q \sim r$, which contradicts the fact that $q \in X \setminus S$. Therefore $U \subseteq X \setminus S$, which shows that S is closed. Since X is connected, $S = X$. □

Corollary 30. *Let X be a connected topological space and let $f : X \to Y$ be a continuous map. If every $x \in X$ has a neighborhood U on which f is constant, then f is constant on X.*

Theorem 31. *[Problem 3-1] Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Then $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.*

Proof. Suppose F is constant on each component of M and let $p \in M$. For all $v \in T_pM$ we have $dF_p(v)(f) = v(f \circ F) = 0$ since $f \circ F$ is constant on a neighborhood of p (see Proposition 3.8), so $dF_p = 0$. Conversely, suppose that $dF = 0$, let $p \in M$, and choose smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $\varphi(U)$ is an open ball and $F(U) \subseteq V$. Let $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ be the coordinate representation of F. From equation (3.10), the derivative of $\widehat{F}$ is zero since $dF_p = 0$ and therefore $\widehat{F}$ is constant on $\varphi(U)$. This implies that $F = \psi^{-1} \circ \widehat{F} \circ \varphi$ is constant on U. By Corollary 30, F is constant on each component of M. □

Theorem 32. *[Problem 3-2] Let $M_1, \dots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map*

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. It is clear that α is a linear map. For each $j = 1, \dots, k$, define $e_j : M_j \to M_1 \times \cdots \times M_k$ by $e_j(x) = (p_1, \dots, x, \dots, p_k)$ where x is in the jth position. Each e_j induces a linear map $d(e_j)_{p_j} : T_{p_j}M_j \to T_p(M_1 \times \cdots \times M_k)$, so we can define a linear map

$$\begin{aligned} \beta : T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k &\to T_p(M_1 \times \cdots \times M_k) \\ (v_1, \dots, v_k) &\mapsto d(e_1)_{p_1}(v_1) + \cdots + d(e_k)_{p_k}(v_k). \end{aligned}$$

The jth component of $(\alpha \circ \beta)(v_1, \dots, v_k)$ is then given by

$$\begin{aligned} d(\pi_j)_p \left(\sum_{i=1}^k d(e_i)_{p_i}(v_i) \right) &= \sum_{i=1}^k d(\pi_j \circ e_i)_{p_i}(v_i) \\ &= d(\pi_j \circ e_j)_{p_j}(v_j) \\ &= v_j \end{aligned}$$

since $\pi_j \circ e_i$ is constant when $i \neq j$, and $\pi_j \circ e_j = \mathrm{Id}_{M_j}$. This shows that α is surjective. But

$$\dim(T_p(M_1 \times \cdots \times M_k)) = \dim(T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k),$$

so α is an isomorphism. □

Theorem 33. *If (U_j, φ_j) are smooth charts containing p_j as defined in Theorem 32, then $\varphi = \varphi_1 \times \cdots \times \varphi_k$ is a smooth coordinate map containing p. Let n_j be the dimension of M_j, and write $\varphi_j = (x_j^1, \dots, x_j^{n_j})$ for each j so that*

$$\varphi = \varphi_1 \times \cdots \times \varphi_k = (\widetilde{x}_1^1, \dots, \widetilde{x}_1^{n_1}, \dots, \widetilde{x}_k^1, \dots, \widetilde{x}_k^{n_k})$$

where $\widetilde{x}_j^i = x_j^i \circ \pi_j$.

(1) *The collection*

$$\left\{ \left. \frac{\partial}{\partial \widetilde{x}_j^i} \right|_p : 1 \leq j \leq k, 1 \leq i \leq n_j \right\}$$

forms a basis for $T_p(M_1 \times \cdots \times M_k)$.

(2) *Let $v \in T_p(M_1 \times \cdots \times M_k)$ and write $v = \sum_{i,j} v^{ji}\, \partial/\partial \widetilde{x}_j^i\big|_p$. Then the jth component of $\alpha(v)$ is $\sum_i v^{ji}\, \partial/\partial x_j^i\big|_{p_j}$ (the variable j is fixed).*

Proof. Since $\mathcal{B}_j = \left\{ \partial/\partial x_j^1\big|_{p_j}, \dots, \partial/\partial x_j^{n_j}\big|_{p_j} \right\}$ is a basis for $T_{p_j}M_j$, the set $\alpha^{-1}(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k) = \beta(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k)$ is a basis for $T_p(M_1 \times \cdots \times M_k)$. But

$$\begin{aligned} \beta\left(\left. \frac{\partial}{\partial x_j^i} \right|_{p_j} \right)(f) &= d(e_j)_{p_j} \left(\left. \frac{\partial}{\partial x_j^i} \right|_{p_j} \right)(f) \\ &= d(e_j \circ \varphi_j^{-1})_{\varphi_j(p_j)} \left(\left. \frac{\partial}{\partial x_j^i} \right|_{\varphi_j(p_j)} \right)(f) \\ &= \frac{\partial (f \circ e_j \circ \varphi_j^{-1})}{\partial x_j^i}(\varphi_j(p_j)) \\ &= \frac{\partial (f \circ \varphi^{-1})}{\partial \widetilde{x}_j^i}(\varphi(p)) \end{aligned}$$

$$= d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial \widetilde{x}^i_j}\bigg|_{\varphi(p)} \right)(f)$$
$$= \frac{\partial}{\partial \widetilde{x}^i_j}\bigg|_p (f),$$

which proves (1). For (2), we compute the jth component of $\alpha(v)$ as

$$\begin{aligned} d(\pi_j)_p \left(\sum_{r,s} v^{sr} \frac{\partial}{\partial \widetilde{x}^r_s}\bigg|_p \right)(f) &= \sum_{r,s} v^{sr} d(\pi_j)_p \left(\frac{\partial}{\partial \widetilde{x}^r_s}\bigg|_p \right)(f) \\ &= \sum_r v^{jr} \frac{\partial (f \circ \pi_j \circ \varphi^{-1})}{\partial \widetilde{x}^r_j}(\varphi(p)) \\ &= \sum_r v^{jr} \frac{\partial (f \circ \varphi_j^{-1})}{\partial x^r_j}(\varphi_j(p_j)) \\ &= \sum_i v^{ji} \left(\frac{\partial}{\partial x^i_j}\bigg|_{p_j} \right)(f). \end{aligned}$$

□

Theorem 34. *[Problem 3-3] If M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.*

Proof. Let m and n denote the dimensions of M and N respectively. For each $(p, q) \in M \times N$, there is an isomorphism $\alpha_{(p,q)} : T_{(p,q)}(M \times N) \to T_pM \oplus T_qN$ as in Theorem 32. Define a bijection $F : T(M \times N) \to TM \times TN$ by sending each tangent vector $v \in T_{(p,q)}(M \times N)$ to $\alpha_{(p,q)}(v)$. For any $(p, q) \in M \times N$, choose smooth charts (U, φ) containing p and (V, ψ) containing q. Let $\pi : T(M \times N) \to M \times N$ be the natural projection map. Then $\rho = \varphi \times \psi$ is a smooth coordinate map for $M \times N$, $\widetilde{\rho} : \pi^{-1}(U \times V) \to \mathbb{R}^{2(m+n)}$ is a smooth coordinate map for $T(M\times N)$, and $\widetilde{\varphi}\times\widetilde{\psi} : \pi^{-1}(U)\times\pi^{-1}(V) \to \mathbb{R}^{2(m+n)}$ is a smooth coordinate map for $TM \times TN$. By Theorem 33 the coordinate representation of F is

$$\begin{aligned} &F(x^1, \ldots, x^m, y^1, \ldots, y^n, v^1, \ldots, v^{m+n}) \\ &\quad = (x^1, \ldots, x^m, v^1, \ldots, v^m, y^1, \ldots, y^n, v^{m+1}, \ldots, v^{m+n}) \end{aligned}$$

which is clearly smooth. Similarly, F^{-1} is also smooth. □

Theorem 35. *[Problem 3-4] $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.*

Proof. We begin by showing that for each $z \in \mathbb{S}^1$ we can choose an associated tangent vector $v_z \in T_z\mathbb{S}^1$. Let $\theta : U \to \mathbb{R}$ be any angle function defined on a neighborhood of

z, and set $v_z = d/dx|_z = (d\theta)_z^{-1}\, d/dx|_{\theta(z)}$. If $\theta' : U' \to \mathbb{R}$ is any other angle function defined around z then $\theta - \theta'$ is constant on $U \cap U'$ and $(\theta' \circ \theta^{-1})(t) = t + c$ for some $c \in \mathbb{R}$. Therefore $(d\theta')_z^{-1}\, d/dx|_{\theta'(z)} = (d\theta')_z^{-1}\, d/dx'|_{\theta'(z)}$, and v_z is well-defined.

Define a map $F : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$ by sending (z, r) to rv_z. If $\theta : U \to \mathbb{R}$ is an angle function defined around z then the coordinate representation of F is $F(z, r) = (\theta(z), r)$, so F is smooth. An inverse to F is defined by sending $v \in T_z\mathbb{S}^1$ to (z, r), where $v = rv_z$ (since $T\mathbb{S}^1$ is one-dimensional). The coordinate representation of F^{-1} is $F^{-1}(x, r) = (\theta^{-1}(x), r)$, so F^{-1} is also smooth. □

Theorem 36. *[Problem 3-5] Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. There is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no* diffeomorphism *with the same property.*

Proof. The map defined by

$$F(x, y) = \frac{\sqrt{x^2 + y^2}}{\max(|x|, |y|)}(x, y)$$

is a homeomorphism such that $F(\mathbb{S}^1) = K$. Suppose there is a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. Let $\gamma : (-1, 1) \to \mathbb{S}^1$ be given by $s \mapsto \exp(2\pi i(s + c))$, where c is chosen so that $(F \circ \gamma)(0) = (1, 1)$. We can assume without loss of generality that $(F \circ \gamma)((-\varepsilon, 0))$ lies on the right edge of K and that $(F \circ \gamma)((0, \varepsilon))$ lies on the top edge of K, for small ε. Then there are smooth functions γ_1, γ_2 such that $(F \circ \gamma)(t) = (1, \gamma_1(t))$ for $t \in (-\varepsilon, 0)$ and $(F \circ \gamma)(t) = (\gamma_2(t), 1)$ for $t \in (0, \varepsilon)$. Since $(F \circ \gamma)'(t) = (0, \gamma_1'(t))$ and $(F \circ \gamma)'(t) = (\gamma_2'(t), 0)$, taking $t \to 0$ shows that $(F \circ \gamma)'(0) = 0$. But $dF(\gamma'(0)) \neq 0$ since $\gamma'(0) \neq 0$ and dF is an isomorphism, so this is a contradiction. □

Theorem 37. *[Problem 3-7] Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\phi : \mathcal{D}_pM \to T_pM$ by $(\phi v)f = v([f]_p)$. Then ϕ is an isomorphism.*

Proof. It is clear that ϕ is a homomorphism. If $\phi v = 0$ then $v([f]_p) = 0$ for all $f \in C^\infty(M)$, so $v = 0$ and therefore ϕ is injective. If $v \in T_pM$ then we can define $v' \in \mathcal{D}_pM$ by setting $v'([f]_p) = v(f)$. This is well-defined due to Proposition 3.8. Then $\phi v' = v$, so ϕ is surjective. □

Theorem 38. *[Problem 3-8] Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. The map $\psi : \mathcal{V}_pM \to T_pM$ defined by $\psi[\gamma] = \gamma'(0)$ is well-defined and bijective.*

Proof. If $\gamma_1, \gamma_2 \in [\gamma]$ then $\gamma_1'(0)(f) = (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) = \gamma_2'(0)(f)$, which shows that ψ is well-defined. The same argument shows that ψ is injective. Finally, Proposition 3.23 shows that ψ is surjective. $\square$

Chapter 4. Submersions, Immersions, and Embeddings

Theorem 39. *[Exercise 4.4]*

(1) *The composition of two smooth submersions is a smooth submersion.*
(2) *The composition of two smooth immersions is a smooth immersion.*
(3) *The composition of two maps of constant rank need not have constant rank.*

Proof. Let $F : M \to N$ and $G : N \to P$ be smooth maps; parts (1) and (2) follow from the fact that $d(G \circ F)_x = dG_{F(x)} \circ dF_x$. For (3), define $F : \mathbb{R} \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x) = (1, x), \quad G(x, y) = x + y^2$$

so that

$$DF(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad DG(x, y) = \begin{bmatrix} 1 & 2y \end{bmatrix}, \quad D(G \circ F)(x) = 2x.$$

$\square$

Theorem 40. *[Exercise 4.7]*

(1) *Every composition of local diffeomorphisms is a local diffeomorphism.*
(2) *Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.*
(3) *Every local diffeomorphism is a local homeomorphism and an open map.*
(4) *The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.*
(5) *Every diffeomorphism is a local diffeomorphism.*
(6) *Every bijective local diffeomorphism is a diffeomorphism.*
(7) *A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.*

Proof. Let $F : M \to N$ and $G : N \to P$ be local diffeomorphisms. If $x \in M$ then there are neighborhoods U of x and V of $F(x)$ such that $F|_U : U \to V$ is a diffeomorphism, and there are neighborhoods V' of $F(x)$ and W of $(G \circ F)(x)$ such that $G|_{V'} : V' \to W$ is a diffeomorphism. Then $(G \circ F)|_{F^{-1}(V \cap V')}$ is a diffeomorphism onto its image, which proves (1). Parts (2)-(5) are obvious. Suppose $F : M \to N$ is a bijective local diffeomorphism. If $y \in N$ then there are neighborhoods U of $F^{-1}(y)$ and V of y such that $F|_U : U \to V$ is a diffeomorphism, so $(F^{-1})|_V$ is smooth. Theorem 11 then shows that F^{-1} is smooth, which proves (6). Part (7) is obvious. $\square$

Theorem 41. *[Exercise 4.10] Suppose M, N, P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism.*

(1) *If $G : P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.*
(2) *If in addition F is surjective and $G : N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.*

Proof. For (1), if $F \circ G$ is smooth and $x \in P$ then there are neighborhoods U of $G(x)$ and V of $(F \circ G)(x)$ such that $F|_U : U \to V$ is a diffeomorphism, so $G|_{G^{-1}(U)} = (F|_U)^{-1} \circ (F \circ G)|_{G^{-1}(U)}$ is smooth. By Theorem 11, G is smooth. For (2), if $G \circ F$ is smooth and $x \in N$ then there is a point $p \in M$ such that $x = F(p)$, and there are neighborhoods U of p and V of x such that $F|_U : U \to V$ is a diffeomorphism. Then $G|_V = G \circ F \circ (F|_U)^{-1}$ is smooth, so G is smooth by Theorem 11. $\square$

Example 42. [Exercise 4.24] Give an example of a smooth embedding that is neither an open nor a closed map.

Let $X = [0, 1)$ and $Y = [-1, 1]$, and let $f : X \to Y$ be the identity map on X. Then f is a smooth embedding, X is both open and closed, but $f(X)$ is neither open nor closed.

Example 43. [Exercise 4.27] Give an example of a smooth map that is a topological submersion but not a smooth submersion.

Consider $f(x) = x^3$ at $x = 0$.

Theorem 44. *[Exercise 4.32] Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are surjective submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.*

Proof. Theorem 4.30 shows that there are unique smooth maps $\widetilde{\pi}_1 : N_2 \to N_1$ and $\widetilde{\pi}_2 : N_1 \to N_2$ such that $\pi_1 = \widetilde{\pi}_1 \circ \pi_2$ and $\pi_2 = \widetilde{\pi}_2 \circ \pi_1$. Then $\widetilde{\pi}_2 \circ \widetilde{\pi}_1 \circ \pi_2 = \widetilde{\pi}_2 \circ \pi_1 = \pi_2$, so $\widetilde{\pi}_2 \circ \widetilde{\pi}_1 = \mathrm{Id}_{N_2}$ by the uniqueness part of Theorem 4.30. Similarly, $\widetilde{\pi}_1 \circ \widetilde{\pi}_2 = \mathrm{Id}_{N_1}$. $\square$

Theorem 45. *[Exercise 4.33]*

(1) *Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.*
(2) *An injective smooth covering map is a diffeomorphism.*
(3) *A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.*

Proof. Let $\pi : E \to M$ be a smooth covering map. Let $e \in E$ and $x = \pi(e)$, let U be an evenly covered neighborhood of x, and let $\widetilde{U}$ be the component of $\pi^{-1}(U)$ containing

e. By definition $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is a diffeomorphism, so π is a local diffeomorphism and the other properties in (1) follow. If π is injective then π is a bijection. Every bijective local diffeomorphism is a diffeomorphism, proving (2). Part (3) is obvious. □

Theorem 46. *[Exercise 4.37] Suppose $\pi : E \to M$ is a smooth covering map. Every local section of π is smooth.*

Proof. Let $\tau : V \to E$ be a local section of π where V is open in M. Let $x \in V$ and let U be an evenly covered neighborhood of x contained in V. By Proposition 4.36, there is a unique smooth local section $\sigma : U \to E$ such that $\sigma(x) = \tau(x)$. Therefore $\tau|_U = \sigma$, and τ is smooth by Theorem 11. □

Theorem 47. *[Exercise 4.38] Suppose $E_1, \dots, E_k$ and $M_1, \dots, M_k$ are smooth manifolds (without boundary), and $\pi_i : E_i \to M_i$ is a smooth covering map for each $i = 1, \dots, k$. Then $\pi_1 \times \cdots \times \pi_k : E_1 \times \cdots \times E_k \to M_1 \times \cdots \times M_k$ is a smooth covering map.*

Proof. This is true for topological covering maps. But any finite product of local diffeomorphisms is a local diffeomorphism, so the result follows from Theorem 45. □

Theorem 48. *Suppose $\pi : E \to M$ is a smooth covering map. If $U \subseteq M$ is evenly covered in the topological sense, then π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U.*

Proof. Let $\widetilde{U}$ be a component of $\pi^{-1}(U)$ and suppose $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism (or merely a bijection). Since $\pi|_{\widetilde{U}}$ is a local diffeomorphism and it is bijective, it is a diffeomorphism. □

Theorem 49. *[Exercise 4.42] Suppose M is a connected smooth n-manifold with boundary, and $\pi : E \to M$ is a topological covering map. Then E is a topological n-manifold with boundary such that $\partial E = \pi^{-1}(\partial M)$, and it has a unique smooth structure such that π is a smooth covering map.*

Proof. Identical to Proposition 4.40. □

Theorem 50. *[Exercise 4.44] If M is a connected smooth manifold, there exists a simply connected smooth manifold $\widetilde{M}$, called the universal covering manifold of M, and a smooth covering map $\pi : \widetilde{M} \to M$. The universal covering manifold is unique in the following sense: if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi' : \widetilde{M}' \to M$, then there exists a diffeomorphism $\Phi : \widetilde{M} \to \widetilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. Every manifold has a universal covering space, so let $\pi : \widetilde{M} \to M$ be a universal covering. Proposition 4.40 shows that $\widetilde{M}$ is a smooth manifold with a unique smooth structure such that π is a smooth covering map. If $\pi' : \widetilde{M}' \to M$ is another universal smooth covering, then there is a homeomorphism $\Phi : \widetilde{M} \to \widetilde{M}'$ such that $\pi' \circ \Phi = \pi$. Let $e \in \widetilde{M}$, let $x = \pi(e)$ and choose some $e' \in \pi^{-1}(\{x\})$. Let U be an evenly covered neighborhood of x with respect to both π and π', let $\widetilde{U}$ be the component of $\pi^{-1}(U)$ containing e, and let $\widetilde{U}'$ be the component of $(\pi')^{-1}(U)$ containing e'. Then $\Phi|_{\widetilde{U}} = (\pi'|_{\widetilde{U}'})^{-1} \circ \pi|_{\widetilde{U}}$, so $\Phi|_{\widetilde{U}}$ is a diffeomorphism. This shows that Φ is a bijective local diffeomorphism and therefore Φ is a diffeomorphism. □

Example 51. [Problem 4-1] Use the inclusion map $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary.

Lemma 3.11 shows that $d\iota_0$ is invertible. If Theorem 4.5 holds, then there are neighborhoods $U \subseteq \mathbb{H}^n$ of 0 and a neighborhood $V \subseteq \mathbb{R}^n$ of 0 such that $\iota|_U \to V$ is a diffeomorphism. Since ι is a bijection, we must have $U = V$. This is impossible, because U cannot be open in $\mathbb{R}^n$.

Theorem 52. *[Problem 4-2] Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \to N$ is smooth. If $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \operatorname{Int} N$.*

Proof. Let n be the dimension of M, which is the same as the dimension of N. Suppose $F(p) \in \partial N$. Choose smooth charts (U, φ) centered at p and (V, ψ) centered at $F(p)$ such that $\psi(V) \subseteq \mathbb{H}^n$, $F(U) \subseteq V$ and $\psi(F(p)) = 0$. Let $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$. Then $d\widetilde{F}_0$ is invertible, so the inverse function theorem shows that there are neighborhoods $\widetilde{U}_0$ of 0 and $\widetilde{V}_0$ of 0 open in $\mathbb{R}^n$ such that $\widetilde{F}|_{\widetilde{U}_0} : \widetilde{U}_0 \to \widetilde{V}_0$ is a diffeomorphism. But $\widetilde{V}_0$ is a neighborhood of 0 open in $\mathbb{R}^n$ and contained in $\mathbb{H}^n$, which is impossible. □

Example 53. [Problem 4-4] Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. The image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

Let $(z_1, z_2) \in \mathbb{T}^2$ and choose $s_1, s_2 \in \mathbb{R}$ so that $e^{2\pi i s_1} = z_1$ and $e^{2\pi i s_2} = z_2$. Then $\gamma(s_1 + n) = (e^{2\pi i s_1}, e^{2\pi i \alpha s_1} e^{2\pi i \alpha n})$, so it remains to show that

$$S = \left\{ e^{2\pi i \alpha n} : n \in \mathbb{Z} \right\}$$

is dense in $\mathbb{S}^1$ whenever α is irrational. Let $\varepsilon > 0$. By Lemma 4.21, there exist integers n, m such that $|n\alpha - m| < \varepsilon$. Let $\theta = n\alpha - m$ so that

$$e^{2\pi i \theta} = e^{2\pi i \alpha n} \in S$$

and $\theta \neq 0$ since α is irrational. Any integer power of $e^{2\pi i \theta}$ is also a member of S, and since ε was arbitrary, we can approximate any point on $\mathbb{S}^1$ by taking powers of $e^{2\pi i \theta}$.

Example 54. We identify $\mathbb{S}^3$ with the set $\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. Let $p : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$ be the inverse of the stereographic projection given by

$$p(x+yi) = \frac{1}{x^2+y^2+1}(2x, 2y, x^2+y^2-1),$$

or

$$p(z) = \frac{1}{|z|^2+1}(z+\bar{z}, -i(z-\overline{z}), |z|^2-1).$$

Define $q : \mathbb{S}^3 \setminus \{\mathbb{S}^1 \times \{0\}\} \to \mathbb{C}$ by $(z,w) = z/w$. Then we have a smooth map $p \circ q : \mathbb{S}^3 \setminus \{\mathbb{S}^1 \times \{0\}\} \to \mathbb{S}^2 \setminus \{N\}$ given by

$$\begin{aligned}(p \circ q)(z,w) &= \frac{1}{|z/w|^2+1}\left(\frac{z}{w} + \frac{\bar{z}}{\bar{w}}, -i\left(\frac{z}{w} - \frac{\bar{z}}{\bar{w}}\right), \left|\frac{z}{w}\right|^2 - 1\right) \\ &= (z\bar{w} + \bar{z}w, -i(z\bar{w} - \bar{z}w), z\bar{z} - w\bar{w}).\end{aligned}$$

It is easy to see that we can extend $p \circ q$ so that it maps $\mathbb{S}^3$ into $\mathbb{S}^2$ by using the formula above.

Theorem 55. *[Problem 4-5] Let $\mathbb{CP}^n$ be n-dimensional complex projective space.*

1. *The quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.*
2. *$\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.*

Proof. Let $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$ and assume without loss of generality that $z_{n+1} \neq 0$. We have a coordinate map φ given by

$$[w_1, \ldots, w_n, w_{n+1}] \mapsto \left(\frac{w_1}{w_{n+1}}, \ldots, \frac{w_n}{w_{n+1}}\right).$$

Then

$$(\varphi \circ \pi)'(w_1, \ldots, w_{n+1}) = \begin{bmatrix} w_{n+1}^{-1} & 0 & \cdots & 0 & -w_1 w_{n+1}^{-2} \\ 0 & w_{n+1}^{-1} & \cdots & 0 & -w_2 w_{n+1}^{-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{n+1}^{-1} & -w_n w_{n+1}^{-2} \end{bmatrix},$$

which clearly has full rank when $w_k = z_k$ for $k = 1, \ldots, n+1$ (since the first n columns form an invertible matrix). This proves (1). The map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ is a surjective smooth submersion. Define a map $\psi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{S}^2$ by

$$\psi(z_1, z_2) = (p \circ q)\left(\frac{(z_1, z_2)}{|(z_1, z_2)|}\right),$$

where $p \circ q$ is the map in 54. By Theorem 4.30, ψ descends to a unique smooth map $\widetilde{\psi} : \mathbb{CP}^1 \to \mathbb{S}^2$ satisfying $\widetilde{\psi} \circ \pi = \psi$. This map is a diffeomorphism. $\square$

Theorem 56. *[Problem 4-6] Let M be a nonempty smooth compact manifold. There is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.*

Proof. By Proposition 4.28, $F(M)$ is open in $\mathbb{R}^k$. Also, $F(M)$ is compact because M is compact. Therefore $F(M)$ is open, closed and bounded. But $\mathbb{R}^k$ is connected and M is nonempty, so $F(M) = \mathbb{R}^k$. This contradicts the boundedness of $F(M)$. □

Theorem 57. *[Problem 4-7] Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. There is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F : \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, then Id_N is a diffeomorphism between N and $\widetilde{N}$.*

Proof. Write ι_N for the identity map from $\widetilde{N}$ to N, and $\iota_{\widetilde{N}}$ for the identity map from N to $\widetilde{N}$. Then ι_N is smooth if and only if $\iota_N \circ \pi = \pi$ is smooth and $\iota_{\widetilde{N}}$ is smooth if and only if $\iota_{\widetilde{N}} \circ \pi = \pi$ is smooth. Both of these conditions hold, and since ι_N and $\iota_{\widetilde{N}}$ are inverses, $\mathrm{Id}_N = \iota_N = \iota_{\widetilde{N}}$ is a diffeomorphism. □

Theorem 58. *[Problem 4-8] Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Then π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion (cf. Theorem 4.29).*

Proof. It is clear that $F \circ \pi$ is smooth if F is smooth. If $F \circ \pi$ is smooth then the map $x \mapsto (F \circ \pi)(x, 1)$ is smooth and identical to F, so F is smooth. However, we can compute

$$D\pi(x, y) = \begin{bmatrix} y & x \end{bmatrix},$$

which does not have full rank when $(x, y) = (0, 0)$. □

Theorem 59. *[Problem 4-9] Let M be a connected smooth manifold and let $\pi : E \to M$ be a topological covering map. There is only one smooth structure on E such that π is a smooth covering map (cf. Proposition 4.40).*

Proof. Let $\mathcal{M}$ be the smooth structure constructed in Proposition 4.40. Suppose there is another smooth structure $\mathcal{M}'$ for which π is a smooth covering map. Let $p \in E$ and let $(U, \widetilde{\varphi})$ be a smooth chart in $\mathcal{M}'$ containing p. Let V be an evenly covered neighborhood of $\widetilde{\varphi}(p)$ and let $\widetilde{V}$ be the component of $\pi^{-1}(V)$ containing p so that $\pi|_{\widetilde{V}} : \widetilde{V} \to V$ is a diffeomorphism (with respect to $\mathcal{M}'$). By shrinking V and $\widetilde{V}$ if necessary, we may assume that there is a smooth chart (V, ψ) containing $\varphi(p)$. Then $\psi \circ \pi|_{\widetilde{V}}$ is a smooth coordinate map in $\mathcal{M}$, and

$$\widetilde{\varphi} \circ (\psi \circ \pi|_{\widetilde{V}})^{-1} = \widetilde{\varphi} \circ (\pi|_{\widetilde{V}})^{-1} \circ \psi^{-1}$$

is smooth. This shows that $\widetilde{\varphi}$ and $\psi \circ \pi|_{\widetilde{V}}$ are smoothly compatible, and since p was arbitrary, $\mathcal{M} = \mathcal{M}'$. □

Example 60. [Problem 4-10] Show that the map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map.

Since $\mathbb{S}^n$ is compact, it is clear that q is proper. For each $i = 1, \ldots, n+1$, let

$$\begin{aligned} S_i^+ &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_i > 0\}, \\ S_i^- &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_i < 0\}. \end{aligned}$$

Then $q|_{S_i^+}$ and $q|_{S_i^-}$ are diffeomorphisms for $i = 1, \ldots, n+1$, which shows that q is a local diffeomorphism. By Proposition 4.46, q is a smooth covering map.

Example 61. [Problem 4-13] Define a map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Let $q : \mathbb{S}^2 \to \mathbb{RP}^2$ be the map of Example 60. Since $F(-x, -y, -z) = F(x, y, z)$, the map F descends to a unique smooth map $\widetilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$. It is easy to check that $\widetilde{F}^{-1}$ is smooth and given by

$$\widetilde{F}^{-1}(t, u, v, w) = q\left(\sqrt{\frac{uv}{w}}, \sqrt{\frac{uw}{v}}, \sqrt{\frac{vw}{u}}\right).$$

Note that we take $uv/w = 0$ if $w = 0$, since $w = yz = 0$ implies that either $u = 0$ or $v = 0$. This is true for the other coordinates. Therefore $\widetilde{F}$ is a homeomorphism onto its image. It remains to show that F is a smooth immersion. Let

$$\begin{aligned} \varphi : \mathbb{RP}^2 \setminus \left\{[x, y, z] \in \mathbb{RP}^2 : z \neq 0\right\} &\to \mathbb{R}^2 \\ [x, y, z] &\mapsto \left(\frac{x}{z}, \frac{y}{z}\right) \end{aligned}$$

be coordinate map. Then

$$\begin{aligned} (\widetilde{F} \circ \varphi^{-1})(x, y) &= \widetilde{F}[x, y, 1] \\ &= (x^2 - y^2, xy, x, y) \end{aligned}$$

so that

$$D(\widetilde{F} \circ \varphi^{-1})(x, y) = \begin{bmatrix} 2x & -2y \\ y & x \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which clearly has full rank. We can perform similar computations for the other coordinate maps, showing that $d\widetilde{F}_{[x,y,z]}$ is always surjective.

Chapter 5. Submanifolds

Example 62. [Exercise 5.10] Show that spherical coordinates (Example C.38) form a slice chart for $\mathbb{S}^2$ in $\mathbb{R}^3$ on any open subset where they are defined.

Spherical coordinates are given by

$$(\rho, \varphi, \theta) \mapsto (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

For the image to be a subset of $\mathbb{S}^2$, we must have $\rho = 1$.

Theorem 63. *[Exercise 5.24] Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold, $\iota : S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X : U \to M$ is a smooth local parametrization of S if and only if there is a smooth coordinate chart (V, φ) for S such that $X = \iota \circ \varphi^{-1}$. Therefore, every point of S is in the image of some local parametrization.*

Proof. If X is a smooth local parametrization of S then $X^{-1} : X(U) \to U$ is a diffeomorphism, so $(X(U), X^{-1})$ is a smooth chart such that $X = \iota \circ (X^{-1})^{-1}$. Conversely, suppose that (V, φ) is a smooth chart such that $X = \iota \circ \varphi^{-1}$. Then $X(\varphi(V)) = V$ is open in S and X is a diffeomorphism onto V, which shows that X is a smooth local parametrization. $\square$

Theorem 64. *[Exercise 5.36] Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.*

Proof. Let $\iota : S \hookrightarrow M$ be the inclusion map. If $v \in T_pS$ then by Proposition 3.23 there is a smooth curve $\gamma_0 : J \to S$ such that $0 \in J$, $\gamma_0(0) = p$ and $\gamma_0'(0) = v$. Therefore $\gamma = \iota \circ \gamma_0$ is the desired smooth curve. Conversely, if there is a smooth curve γ with the specified properties then $\gamma_0 : J \to S$ given by $t \mapsto \gamma(t)$ is smooth. Therefore $\gamma_0'(0) \in T_pS$ and $v = \gamma'(0) = d\iota_p(\gamma_0'(0))$, so v is in T_pS (as a subspace of T_pM). $\square$

Theorem 65. *[Exercise 5.40] Suppose $S \subseteq M$ is a level set of a smooth map $\Phi : M \to N$ with constant rank. Then $T_pS = \ker d\Phi_p$ for each $p \in S$.*

Proof. Follows from Theorem 5.12 and Proposition 5.38. $\square$

Theorem 66. *[Exercise 5.42] Suppose M is a smooth n-dimensional manifold with boundary, $p \in \partial M$, and (x^i) are any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to ∂M are those with zero x^n-component. Thus, T_pM is the disjoint union of $T_p\partial M$, the set of inward-pointing vectors, and the set of outward-pointing vectors, and $v \in T_pM$ is inward-pointing if and only if $-v$ is outward-pointing.*

Proof. As in Proposition 3.23. □

Theorem 67. *[Exercise 5.44] Suppose M is a smooth manifold with boundary, f is a boundary defining function, and $p \in \partial M$. A vector $v \in T_pM$ is inward-pointing if and only if $vf > 0$, outward-pointing if and only if $vf < 0$, and tangent to ∂M if and only if $vf = 0$.*

Proof. Let $(x^1, \ldots, x^n)$ be smooth coordinates defined on a neighborhood of p. Since f is a boundary defining function, $(\partial f/\partial x^i)(p) = 0$ for $i = 1, \ldots, n-1$ and $(\partial f/\partial x^n)(p) > 0$. Write $v = \sum_{i=1}^n v^i(\partial/\partial x^i|_p)$ so that

$$vf = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p) = v^n \frac{\partial f}{\partial x^n}(p);$$

then the result follows from Theorem 66. □

Example 68. [Problem 5-1] Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1)$ is a regular value of Φ, and that the level set $S = \Phi^{-1}(\{(0, 1)\})$ is diffeomorphic to $\mathbb{S}^2$.

We compute

$$D\Phi(x, y, s, t) = \begin{bmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{bmatrix}.$$

This matrix clearly has full rank when $s \neq 0$ or $t \neq 0$, so suppose that $s = t = 0$. The determinant of the submatrix formed by the first two columns is

$$2x(2y + 1) - 2x = 4xy,$$

which is zero if and only if $x = 0$ or $y = 0$. In either case, we cannot have $\Phi(x, y, s, t) = (0, 1)$. This shows that $d\Phi_p$ is surjective for all $p \in S$. Define $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\varphi(x, y, s, t) = (x, x^2 + y, s, t);$$

then φ is a diffeomorphism and $\widetilde{\Phi} = (\Phi \circ \varphi^{-1})(u, v, s, t) = (v, v + (v - u^2)^2 + s^2 + t^2)$. It is easy to see that

$$\varphi(S) = \left\{(u, 0, s, t) \in \mathbb{R}^4 : u^4 + s^2 + t^2 = 1\right\}.$$

This set can be considered as an embedded submanifold of $\mathbb{R}^4$ diffeomorphic to $\mathbb{S}^2$, such that $\varphi|_S$ is a diffeomorphism. Therefore S is diffeomorphic to $\varphi(S)$.

Theorem 69. *[Problem 5-2] If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.*

Proof. Let $x \in \partial M$ and choose a smooth boundary chart (U, φ) containing x. Then

$$\varphi(\partial M \cap U) = \left\{(x^1, \ldots, x^n) \in \varphi(U) : x^n = 0\right\},$$

which shows that ∂M satisfies the local $(n-1)$-slice condition. The result follows from Theorem 5.8 and Proposition 5.5. □

Theorem 70. *[Problem 5-3] Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.*

1. *S has codimension 0 in M.*
2. *The inclusion map $S \subseteq M$ is proper.*
3. *S is compact.*

Proof. If (1) holds then Theorem 4.5 shows that the inclusion map $\iota : S \hookrightarrow M$ is an open map. The result follows from Proposition 4.22. □

Example 71. [Problem 5-4] Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 given by

$$\beta(t) = (\sin 2t, \sin t)$$

is not an embedded submanifold of $\mathbb{R}^2$.

It is well-known that the image of β with the subset topology is not a topological manifold at all. (This can be seen by considering the point $(0, 0)$, around which there is no coordinate map into $\mathbb{R}$.)

Example 72. [Problem 5-5] Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that $\gamma(\mathbb{R})$ is not an embedded submanifold of the torus.

Again, $\gamma(\mathbb{R})$ with the subset topology is not a topological manifold because it is not locally Euclidean.

Theorem 73. *[Problem 5-6] Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:*

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M,\ v \in T_x M,\ |v| = 1\}.$$

*It is called the **unit tangent bundle of** M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Let $(x, v) \in UM$. Since M is an embedded submanifold of $\mathbb{R}^n$, we can choose a smooth chart (U, φ) for $\mathbb{R}^n$ containing x such that

$$\varphi(M \cap U) = \left\{(x^1, \ldots, x^n) \in \varphi(U) : x^{m+1} = \cdots = x^n = 0\right\}.$$

Similarly, $\mathbb{S}^{m-1}$ is an embedded submanifold of $\mathbb{R}^m$, so we can choose a smooth chart (V,ψ) for $\mathbb{R}^m$ containing v such that

$$\psi(\mathbb{S}^{m-1}\cap V)=\left\{(x^1,\dots,x^m)\in\psi(V):x^m=0\right\}.$$

Write $\varphi=(x^1,\dots,x^n)$ and $\widetilde{U}=\varphi^{-1}(U)$. By definition, the map $\widetilde{\varphi}:\widetilde{U}\to\mathbb{R}^{2n}$ given by

$$v^i\left.\frac{\partial}{\partial x^i}\right|_p\mapsto(x^1(p),\dots,x^n(p),v^1,\dots,v^n)$$

is a coordinate map for $T\mathbb{R}^n$. By shrinking $\widetilde{U}$, we can assume that $V\subseteq\pi(\widetilde{\varphi}(\widetilde{U}))$, where $\pi:\mathbb{R}^{2n}\to\mathbb{R}^m$ is the projection onto the coordinates $n+1,\dots,n+m$. Define

$$\theta(x^1,\dots,x^n,v^1,\dots,v^n)=(x^1,\dots,x^n,\psi(v^1,\dots,v^m),v^{m+1},\dots,v^n);$$

then θ is a diffeomorphism onto its image, and $\theta\circ\widetilde{\varphi}:\widetilde{U}\to\mathbb{R}^{2n}$ is still a coordinate map for $T\mathbb{R}^n$. Furthermore,

$$\begin{aligned}&(\theta\circ\widetilde{\varphi})(UM\cap\widetilde{U})\\&=\left\{(x^1,\dots,x^n,v^1,\dots,v^n)\in(\theta\circ\widetilde{\varphi})(\widetilde{U}):x^{m+1}=\dots=x^n=v^m=\dots=v^n=0\right\},\end{aligned}$$

so UM satisfies the local $(2m-1)$-slice condition. By Theorem 5.8, UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n$. □

Example 74. [Problem 5-7] Let $F:\mathbb{R}^2\to\mathbb{R}$ be defined by $F(x,y)=x^3+xy+y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

We compute

$$DF(x,y)=\begin{bmatrix}3x^2+y & x+3y^2\end{bmatrix},$$

which has full rank unless $3x^2+y=0$ and $x+3y^2=0$. In this case, we have

$$y=-3x^2=-3(-3y^2)^2=-27y^4$$

and

$$0=y+27y^4=y(3y+1)(9y^2-3y+1),$$

so $y=0$ or $y=-\frac{1}{3}$. By symmetry, we have $x=0$ or $x=-\frac{1}{3}$. From this, it is clear that $DF(x,y)$ has full rank if and only if $(x,y)\neq(0,0)$ and $(x,y)\neq(-\frac{1}{3},-\frac{1}{3})$. Note that $F(0,0)=0$ and $F(-\frac{1}{3},-\frac{1}{3})=\frac{1}{27}$. We first examine the level set $S_1=F^{-1}(\{0\})$. The point $p=(0,0)$ is a saddle point of F, so S_1 is a self-intersecting curve at p. Next, we examine the level set $S_2=F^{-1}(\left\{\frac{1}{27}\right\})$. Since F has a strict local maximum at $p=(-\frac{1}{3},-\frac{1}{3})$, the point p is isolated in S_2. In both cases, the level sets fail to be locally Euclidean. Therefore, we can conclude that $F^{-1}(\{c\})$ is an embedded 1-dimensional submanifold of $\mathbb{R}^2$ if and only if $c\neq0$ and $c\neq\frac{1}{27}$.

Theorem 75. *Any closed ball in $\mathbb{R}^n$ is an n-dimensional manifold with boundary.*

Proof. Let $\overline{\mathbb{B}^n} = \overline{B_1(0)}$ be the closed unit ball in $\mathbb{R}^n$, let $N = (0, \ldots, 0, 1)$ be the "north pole", and let $T = \{0\}^{n-1} \times [0, 1]$ be the vertical line connecting 0 and N. Define $\sigma : \overline{\mathbb{B}^n} \setminus T \to \mathbb{H}^n$ given by

$$\sigma(x_1, \ldots, x_n) = \left(\frac{x_1}{\|x\| - x_n}, \ldots, \frac{x_{n-1}}{\|x\| - x_n}, \|x\|^{-1} - 1 \right),$$

with its inverse given by

$$\sigma^{-1}(u_1, \ldots, u_n) = \frac{1}{(u_n + 1)(\|\widetilde{u}\|^2 + 1)}(2u_1, \ldots, 2u_{n-1}, \|\widetilde{u}\|^2 - 1)$$

where $\widetilde{u} = (u_1, \ldots, u_{n-1})$. Since both maps are continuous, this proves that $\overline{\mathbb{B}^n} \setminus T$ is homeomorphic to $\mathbb{H}^n$. Furthermore, we can repeat the same argument by taking the vertical line $T = \{0\}^{n-1} \times [-1, 0]$ instead, giving us Euclidean neighborhoods of every point in $\overline{\mathbb{B}^n}$ except for 0. But the identity map on the open unit ball $\mathbb{B}^n$ is a suitable coordinate chart around 0, so this proves that $\overline{\mathbb{B}^n}$ is an n-manifold with boundary. □

Theorem 76. *[Problem 5-8] Suppose M is a smooth n-manifold and $B \subseteq M$ is a regular coordinate ball. Then $M \setminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$.*

Proof. We have coordinate balls around every point of $M \setminus \overline{B}$, so it remains to show that we have coordinate half-balls around each point of ∂B. Since B is a regular coordinate ball, this reduces to showing that $B_r(0) \setminus B_s(0)$ is a smooth manifold with boundary whenever $s < r$. But this follows easily from the stereographic projection described in Theorem 75. □

Theorem 77. *[Problem 5-9] Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Theorem 36). Then S does not have a topology and smooth structure in which it is an immersed submanifold of $\mathbb{R}^2$.*

Proof. Suppose S has a topology and smooth structure for which the inclusion map $\iota : S \hookrightarrow \mathbb{R}^2$ is a smooth immersion. By Theorem 4.25, there is a neighborhood U of $(1, 1)$ such that $\iota|_U$ is a smooth embedding. We can then apply the argument of Theorem 36 to derive a contradiction. □

Example 78. [Problem 5-10] For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{ (x, y) : y^2 = x(x - 1)(x - a) \right\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold? Let $F(x, y) = y^2 - x(x - 1)(x - a)$; then

$$DF(x, y) = \begin{bmatrix} -(3x^2 - (2a + 2)x + a) & 2y \end{bmatrix}.$$

Therefore 0 is a regular value of F unless there is a point (x,y) such that $y=0$, $3x^2-(2a+2)x+a=0$ and $F(x,y)=-x(x-1)(x-a)=0$. We have the following cases to consider:

$$a=0 \text{ and } (x,y)=(0,0),$$
$$a=1 \text{ and } (x,y)=(1,0).$$

When $a=0$ the point $(0,0)$ is a local minimum of F, so $(0,0)$ is an isolated point of M_0. Therefore M_0 cannot be an embedded or immersed submanifold. When $a=1$ the curve M_1 is self-intersecting at $(1,0)$. By giving M_1 an appropriate topology in which it is disconnected, we can make M_1 an immersed submanifold of $\mathbb{R}^2$.

Theorem 79. *[Problem 5-12] Suppose E and M are smooth manifolds with boundary, and $\pi: E\to M$ is a smooth covering map. The restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M.*

Proof. Let F be a connected component of ∂E. Let $x\in\pi(F)$, let U be an evenly covered neighborhood of x, choose e so that $x=\pi(e)$, and let $\widetilde{U}$ be the component of $\pi^{-1}(U)$ containing e. Then $\pi|_{\widetilde{U}}:\widetilde{U}\to U$ is a diffeomorphism. It is clear that $\pi(\widetilde{U}\cap F)=U\cap\partial M$. Since $\widetilde{U}\cap F$ is an embedded submanifold of $\widetilde{U}$ and $\pi(\widetilde{U}\cap F)$ is an embedded submanifold of U by Theorem 5.11, $\pi|_{\widetilde{U}\cap F}$ is a diffeomorphism. This shows that $U\cap\partial M$ is an evenly covered neighborhood of x, and that the image of $\pi|_F$ is open. Let $x\notin\pi(F)$, let U be an evenly covered neighborhood of x, and let $V=U\cap\partial M$. By shrinking V if necessary, we may assume that V is a connected neighborhood of x in $U\cap\partial M$. Suppose there is a $f\in F$ such that $y=\pi(f)\in V$. Let $\widetilde{U}$ be the component of $\pi^{-1}(U)$ containing f and choose $e\in\widetilde{U}$ so that $x=\pi(e)$. Let γ be a path from x to y in V. Then $(\pi|_{\widetilde{U}})^{-1}\circ\gamma$ is a path from e to f in $\widetilde{U}\cap\partial E$, so $e\in F$ since $f\in F$ and F is a connected component of ∂E. This contradicts the fact that $x\notin\pi(F)$, so $V\subseteq\partial M\setminus\pi(F)$. This shows that $\pi(F)$ is both open and closed in ∂M, so it is a component of ∂M. □

Theorem 80. *[Problem 5-14] Suppose M is a smooth manifold and $S\subseteq M$ is an immersed submanifold. For the given topology on S, there is only one smooth structure making S into an immersed submanifold.*

Proof. Let $\iota: S\hookrightarrow M$ be the inclusion map, which is a smooth immersion. Let S' denote S with some other smooth structure such that $\iota': S'\hookrightarrow M$ is a smooth immersion. Since S and S' have the same topology, the maps $\iota: S\to S'$ and $\iota': S'\to S$ are both continuous, and Theorem 5.29 shows that the maps are inverses of each other. Therefore S is diffeomorphic to S'. □

Theorem 81. *[Problem 5-16] If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, then S has only one topology and smooth structure with respect to which it is an immersed submanifold.*

Proof. Almost identical to Theorem 80. □

Theorem 82. *[Problem 5-17] Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.*

(1) *S is embedded only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M.*
(2) *S is properly embedded only if every $f \in C^\infty(S)$ has a smooth extension to all of M.*

Proof. First suppose that S is embedded. Let $p \in S$ and choose a smooth slice chart (U_p, φ_p) for S in M containing p. Then $S \cap U_p$ is closed in U_p, so Proposition 2.25 shows that there is smooth function $f_p : U_p \to \mathbb{R}$ such that $f_p|_{S\cap U_p} = f|_{S\cap U_p}$. Let $\{\psi_p\}_{p\in S}$ be a partition of unity subordinate to $\{U_p\}_{p\in S}$; then

$$\widetilde{f}(x) = \sum_{p\in S} \psi_p(x) f_p(x)$$

is a smooth function defined on $U = \bigcup_{p\in S} U_p$ such that $\widetilde{f}|_S = f$, taking f_p to be zero on $U \setminus \operatorname{supp} f_p$. For (2), Proposition 5.5 shows that S is closed in M. Therefore we have a partition of unity $\{\psi_p\} \cup \{\psi_0\}$ for M subordinate to $\{U_p\} \cup \{M \setminus S\}$, allowing $\widetilde{f}$ to be defined on all of M. □

Theorem 83. *[Problem 5-19] Suppose $S \subseteq M$ is an embedded submanifold and $\gamma : J \to M$ is a smooth curve whose image happens to lie in S. Then $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. This need not be true if S is not embedded.*

Proof. Corollary 5.30 shows that γ is smooth as a map into S. Denote this map by $\gamma_0 : J \to S$ and let $\iota : S \hookrightarrow M$. Then $\gamma = \iota \circ \gamma_0$, so $\gamma'(t) = d\iota_{\gamma(t)}(\gamma_0'(t))$ by Proposition 3.24 and therefore $\gamma'(t) \in T_{\gamma(t)}S \subseteq T_{\gamma(t)}M$. This need not hold if S is merely an immersed submanifold. For example, consider a curve γ that crosses the point of self-intersection in the figure-eight curve β of Example 4.19, such that γ is not continuous as a map into the image of β. □

Theorem 84. *[Problem 5-21] Suppose M is a smooth manifold and $f \in C^\infty(M)$.*

(1) *For each regular value b of f, the sublevel set $f^{-1}((-\infty, b])$ is a regular domain in M.*
(2) *If a and b are two regular values of f with $a < b$, then $f^{-1}([a, b])$ is a regular domain in M.*

Proof. Let m be the dimension of M. It is clear that $f^{-1}\left((-\infty, b)\right)$ satisfies the local m-slice condition. Furthermore, there is a boundary slice chart around each point of $f^{-1}(\{b\})$. We can see this by applying the argument in Corollary 5.14, with the following modification to Theorem 5.12: for each $p \in f^{-1}(\{b\})$, the rank theorem shows that there are smooth charts (U, φ) centered at p and (V, ψ) containing b for which the coordinate representation $\widehat{f}$ of f has the form

$$\widehat{f}(x^1, \ldots, x^m) = x^1.$$

By modifying $\widehat{f}$ and ψ appropriately, U becomes a boundary slice chart for $f^{-1}\left((-\infty, b]\right)$ in M. Theorem 5.51 then shows that $f^{-1}\left((-\infty, b]\right)$ is an embedded submanifold with boundary in M, and it is properly embedded since it is closed in M. Part (2) is similar. $\square$

Theorem 85. *[Problem 5-22] If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.*

Proof. Let $\mathcal{U} = \{(U_p, \varphi_p)\}_{p \in D}$ be a collection of interior or boundary slice charts for D in M. Then $\mathcal{U} \cup \{M \setminus D\}$ covers M, and we can repeat the argument of Proposition 5.43 by constructing functions $\{f_p\}_{p \in D} \cup \{f_0\}$ as follows: set $f_p = -1$ when $p \in \operatorname{Int} D$, set $f_p(x^1, \ldots, x^n) = -x^n$ when $p \in \partial D$, and set $f_0 = 1$ for $f_0 : M \setminus D \to \mathbb{R}$. If D is compact then we can modify f by using the argument of Proposition 2.28, taking a countable open cover of $M \setminus D$ by precompact open subsets. $\square$

Chapter 6. Sard's Theorem

Theorem 86. *[Exercise 6.7] Let M be a smooth manifold with or without boundary. Any countable union of sets of measure zero in M has measure zero.*

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n has measure zero. Let (U, φ) be a smooth chart; we want to show that

$$A \cap U = \bigcup_{n=1}^{\infty} A_n \cap U$$

has measure zero. But this follows from the fact that

$$\varphi\left(\bigcup_{n=1}^{\infty} A_n \cap U\right) = \bigcup_{n=1}^{\infty} \varphi(A_n \cap U)$$

has measure zero (in $\mathbb{R}^n$) since each $\varphi(A_n \cap U)$ has measure zero. $\square$

Theorem 87. *[Problem 6-1] Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. If $\dim M < \dim N$, then $F(M)$ has measure zero in N.*

Proof. Let $n = \dim N$. Define $\widetilde{F} : M \times \mathbb{R}^k \to N$ by $(x, y) \mapsto F(x)$, where $k = \dim N - \dim M$. This map is smooth because it can be written as the composition $\pi \circ (F \times \mathrm{Id}_{\mathbb{R}^k})$, where $\pi : N \times \mathbb{R}^k \to N$ is the canonical projection. It suffices to show that the image of $\widetilde{F}$ has measure zero in N. Let $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ and (V, ψ) be smooth charts for $M \times \mathbb{R}^k$ and N respectively such that $U_1 \subseteq M$ and $U_2 \subseteq \mathbb{R}^k$. Let $U = U_1 \times U_2$ and $\varphi = \varphi_1 \times \varphi_2$. Then

$$\begin{aligned}\psi(\widetilde{F}(U) \cap V) &= \psi(\widetilde{F}(U \cap \widetilde{F}^{-1}(V))) \\ &= (\psi \circ \widetilde{F} \circ \varphi^{-1})(\varphi(U \cap \widetilde{F}^{-1}(V))) \\ &\subseteq (\psi \circ \widetilde{F} \circ \varphi^{-1})(\varphi(U) \cap (\varphi_1(F^{-1}(V)) \times \{0\})),\end{aligned}$$

which has measure zero in $\mathbb{R}^n$ by Proposition 6.5. Therefore $\widetilde{F}(U)$ has measure zero. But $\widetilde{F}(M \times \mathbb{R}^k)$ can be written as a union of countably many sets of this form, so $\widetilde{F}(M \times \mathbb{R}^k) = F(M)$ has measure zero in N. $\square$

Theorem 88. *[Problem 6-2] Every smooth n-manifold (without boundary) admits a smooth immersion into $\mathbb{R}^{2n}$.*

Proof. Let M be a smooth n-manifold; by Theorem 6.15, we can assume that M is an embedded submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subseteq T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M (see Theorem 73) and define $G : UM \to \mathbb{RP}^{2n}$ by $(x, v) \mapsto [v]$. This map is smooth because it is the composition $UM \to \mathbb{R}^{2n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{RP}^{2n}$. Since $\dim UM < \dim \mathbb{RP}^{2n}$, Sard's theorem shows that the image of G has measure zero in $\mathbb{RP}^{2n}$, and since $\pi(\mathbb{R}^{2n})$ has measure zero in $\mathbb{RP}^{2n}$, there is a $v \in \mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G. Let $\pi_v : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the projection with kernel $\mathbb{R}v$; then $\pi_v|_M$ is smooth. Furthermore, if $p \in M$ then $v \notin T_pM$, so $d(\pi_v|_M)_p$ is injective. $\square$

Theorem 89. *[Problem 6-3] Let M be a smooth manifold, let $B \subseteq M$ be a closed subset, and let $\delta : M \to \mathbb{R}$ be a positive continuous function. There is a smooth function $\widetilde{\delta} : M \to \mathbb{R}$ that is zero on B, positive on $M \setminus B$, and satisfies $\widetilde{\delta}(x) < \delta(x)$ everywhere.*

Proof. By Theorem 2.29, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(\{0\}) = B$. By replacing f with $f/(f+1)$, we can assume that $f = 0$ on B and $0 < f \leq 1$ on $M \setminus B$. By Corollary 6.22, there is a smooth $e : M \to \mathbb{R}$ such that $0 < e(x) < \delta(x)$ for all $x \in M$. Then $\widetilde{\delta} = fe$ is the desired function. $\square$

Example 90. [Problem 6-7] By considering the map $F : \mathbb{R} \to \mathbb{H}^2$ given by $F(t) = (t, |t|)$ and the subset $A = [0, \infty) \subseteq \mathbb{R}$, show that the conclusions of Theorem 6.26 and Corollary 6.27 can be false when M has nonempty boundary.

F is smooth on the closed subset $[0, \infty)$, but there is no smooth map $G : \mathbb{R} \to \mathbb{H}^2$ such that $G|_A = F|_A$.

Theorem 91. *[Problem 6-8] Every proper continuous map between smooth manifolds is homotopic to a proper smooth map.*

Proof. Since the retraction $r : U \to M$ is proper, it suffices to show that if F is proper in Theorem 6.21, then $\widetilde{F}$ is also proper. Let $E \subseteq \mathbb{R}^k$ be compact. Since δ is continuous, it attains some maximum $\delta(x_0)$ on E. Let B be a closed ball containing E such that for every $x \in E$, the closed ball of radius $\delta(x)$ around x is contained in B. Since $|F(x) - \widetilde{F}(x)| < \delta(x)$, it is clear that $\widetilde{F}^{-1}(E)$ is a closed subset of $F^{-1}(B)$. But F is proper, so $F^{-1}(B)$ is compact and therefore $\widetilde{F}^{-1}(E)$ is compact. $\square$

Example 92. [Problem 6-9] Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$F(x, y) = (e^y \cos x, e^y \sin x, e^{-y}).$$

For which positive numbers r is F transverse to the sphere $S_r(0) \subseteq \mathbb{R}^3$? For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?

We compute

$$DF(x, y) = \begin{bmatrix} -e^y \sin x & e^y \cos x \\ e^y \cos x & e^y \sin x \\ 0 & -e^{-y} \end{bmatrix}.$$

We have $\|F(x, y)\| = r$ if and only if

$$r^2 = e^{2y} \cos^2 x + e^{2y} \sin^2 x + e^{-2y} = e^{2y} + e^{-2y},$$

i.e.

$$y = \frac{1}{2} \log \left(\frac{1}{2}(r^2 \pm \sqrt{r^4 - 4}) \right).$$

Thus $F^{-1}(S_r(0))$ is always an embedded submanifold of $\mathbb{R}^2$, being either empty ($r < \sqrt{2}$), one line ($r = \sqrt{2}$) or two lines in $\mathbb{R}^2$ ($r > \sqrt{2}$). However, F is transverse to $S_r(0)$ if and only if $r > \sqrt{2}$. Let us examine the case $r = \sqrt{2}$: we have $y = 0$, so the image of $dF_{(x,y)}$ is spanned by the columns of

$$DF(x, 0) = \begin{bmatrix} -\sin x & \cos x \\ \cos x & \sin x \\ 0 & -1 \end{bmatrix}.$$

The corresponding point in $S_{\sqrt{2}}(0)$ is $v(x) = (\cos x, \sin x, 1)$, so the tangent space consists of all vectors orthogonal to $v(x)$. But $v(x)$ is orthogonal to the columns of $DF(x,0)$, so $dF_{(x,y)} + T_{v(x)}S_{\sqrt{2}}(0)$ is a proper subset of $T_{v(x)}\mathbb{R}^3$.

Theorem 93. *[Problem 6-15] Suppose M and N are smooth manifolds and $S \subseteq M \times N$ is an immersed submanifold. Let π_M and π_N denote the projections from $M \times N$ onto M and N, respectively. The following are equivalent:*

(1) *S is the graph of a smooth map $f : M \to N$.*
(2) *$\pi_M|_S$ is a diffeomorphism from S onto M.*
(3) *For each $p \in M$, the submanifolds S and $\{p\} \times N$ intersect transversely in exactly one point.*

If these conditions hold, then S is the graph of the map $f : M \to N$ defined by $f = \pi_N \circ (\pi_M|_S)^{-1}$.

Proof. Let $m = \dim M$ and $n = \dim N$. Suppose $S = \{(x, f(x)) : x \in M\}$ for some smooth map $f : M \to N$. Then S is the image of the map $g : M \to S$ given by $x \mapsto (x, f(x))$, so $\pi_M|_S$ is smooth with inverse g. Conversely, if $\pi_M|_S$ is a diffeomorphism then we can take $f = \pi_N \circ (\pi_M|_S)^{-1}$ in (1). This proves that (1) $\Leftrightarrow$ (2). If (1) holds then $d(\pi_M)_{g(p)} \circ dg_p$ is always nonzero, so $T_pS + T_p(\{p\} \times N) = T_p(M \times N)$. It is also clear that $S \cap (\{p\} \times N)$ consists of exactly one point. If (3) holds then $\pi_M|_S$ is a bijection. For every $p \in M$, the submanifolds S and $\{p\} \times N$ intersect transversely. Since $\dim(\{p\} \times N) = n < \dim(M \times N)$, we must have $\dim d(\pi_M|_S)_p(T_pS) \geq m$. Therefore $d(\pi_M|_S)_p$ is invertible for every $p \in M$, and $\pi_M|_S$ is a diffeomorphism. This proves that (1) $\Leftrightarrow$ (3). □

Chapter 7. Lie Groups

Theorem 94. *[Exercise 7.2] If G is a smooth manifold with a group structure such that the map $t : G \times G \to G$ given by $(g,h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Proof. The inversion map $i(g) = t(e,g)$ is smooth, and so is the multiplication map $m(g,h) = t(g, i(h)) = t(g, t(e,h))$. □

Theorem 95. *[Problem 7-1] For any Lie group G, the multiplication map $m : G \times G \to G$ is a smooth submersion.*

Proof. For each $g \in G$, the map $\sigma_g : G \to G \times G$ given by $x \mapsto (g, g^{-1}x)$ is a smooth local section of m since $m(\sigma_g(x)) = gg^{-1}x = x$. Every point $(g_1, g_2) \in G \times G$ is in the image of σ_{g_1}, so Theorem 4.26 shows that m is a smooth submersion. □

Theorem 96. *[Problem 7-2] Let G be a Lie group.*

(1) *Let* $m : G \times G \to G$ *denote the multiplication map. Using Proposition 3.14 to identify* $T_{(e,e)}(G \times G)$ *with* $T_eG \oplus T_eG$*, the differential* $dm_{(e,e)} : T_eG \oplus T_eG \to T_eG$ *is given by*

$$dm_{(e,e)}(X, Y) = X + Y.$$

(2) *Let* $i : G \to G$ *denote the inversion map. Then* $di_e : T_eG \to T_eG$ *is given by*

$$di_e(X) = -X.$$

Proof. We have

$$\begin{aligned} dm_{(e,e)}(X, Y) &= dm_{(e,e)}(X, 0) + d_{m(e,e)}(0, Y) \\ &= d(m^{(1)})_e(X) + d(m^{(2)})_e(Y), \end{aligned}$$

where $m^{(1)} : G \to G$ is given by $x \mapsto m(x, e)$ and $m^{(2)} : G \to G$ is given by $y \mapsto m(e, y)$. But $m^{(1)} = m^{(2)} = \mathrm{Id}_G$, so $dm_{(e,e)}(X, Y) = X + Y$. This proves (1). Let $n = m \circ p \circ s$ where $s : G \to G \times G$ is given by $x \mapsto (x, x)$ and $p : G \times G \to G \times G$ is given by $(x, y) \mapsto (x, i(y))$; then n is constant, so

$$\begin{aligned} 0 &= dn_e(X) \\ &= dm_{(e,e)}(dp_{(e,e)}(ds_e(X))) \\ &= dm_{(e,e)}(dp_{(e,e)}(X, X)) \\ &= dm_{(e,e)}(X, di_e(X)) \\ &= X + di_e(X). \end{aligned}$$

Therefore $di_e(X) = -X$. □

Theorem 97. *[Problem 7-3] If* G *is a smooth manifold with a group structure such that the multiplication map* $m : G \times G \to G$ *is smooth, then* G *is a Lie group.*

Proof. It suffices to show that the inversion map $i : G \to G$ is smooth. Define $F : G \times G \to G \times G$ by $(g, h) \mapsto (g, gh)$, which is smooth and bijective. We have

$$dF_{(e,e)}(X, Y) = (X, X + Y),$$

so $dF_{(e,e)}$ is an isomorphism. Write L_g and R_g for the translation maps. If $(x, y) \in G \times G$ then it is easy to check that

$$F = ((L_{xy} \circ R_{y^{-1}}) \times \mathrm{Id}_G) \circ F \circ ((L_{y^{-1}x^{-1}} \circ R_y) \times L_{y^{-1}}),$$

so $dF_{(x,y)} = \varphi \circ dF_{(e,e)} \circ \psi$ where φ, ψ are isomorphisms. Therefore $dF_{(x,y)}$ is an isomorphism and F is a bijective local diffeomorphism, i.e. a diffeomorphism. Since $i(g)$ is the second component of $F^{-1}(g, e)$, the inversion map i is smooth. □

Theorem 98. *[Problem 7-4] Let* $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ *denote the determinant function. Then*

$$d(\det)_X(B) = (\det X) \operatorname{tr}(X^{-1}B).$$

Proof. Let $A \in \mathrm{M}(n,\mathbb{R})$ write $A_{i,j}$ for the (i,j) entry of A. Then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) &= \left.\frac{d}{dt}\right|_{t=0} (1 + \operatorname{tr}(A)t + \cdots) \\ &= \operatorname{tr}(A). \end{aligned}$$

If $X \in \mathrm{GL}(n,\mathbb{R})$ and $B \in T_X \mathrm{GL}(n,\mathbb{R})$ we have

$$\det(X + tB) = \det(X)\det(I_n + tX^{-1}B),$$

so

$$\begin{aligned} d(\det)_X(B) &= \left.\frac{d}{dt}\right|_{t=0} \det(X + tB) \\ &= \det(X) \left.\frac{d}{dt}\right|_{t=0} \det(I_n + tX^{-1}B) \\ &= \det(X)\operatorname{tr}(X^{-1}B). \end{aligned}$$

□

Theorem 99. *[Problem 7-5] For any connected Lie group G, the universal covering group is unique in the following sense: if $\widetilde{G}$ and $\widetilde{G}'$ are simply connected Lie groups that admit smooth covering maps $\pi : \widetilde{G} \to G$ and $\pi' : \widetilde{G}' \to G$ that are also Lie group homomorphisms, then there exists a Lie group isomorphism $\Phi : \widetilde{G} \to \widetilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. Let e be the identity in $\widetilde{G}$ and let e' be the identity in $\widetilde{G}'$. By Theorem 50 there is a diffeomorphism $\Phi : \widetilde{G} \to \widetilde{G}'$ such that $\pi' \circ \Phi = \pi$, so it remains to show that Φ is a group homomorphism. By replacing Φ with $\Phi(e)^{-1}\Phi = L_{\Phi(e)^{-1}} \circ \Phi$, we can assume that $\Phi(e) = e'$. (Since $\pi'(\Phi(e)^{-1}\Phi(g)) = \pi'(\Phi(e))^{-1}\pi(g) = \pi(g)$, we still have $\pi' \circ \Phi = \pi$.) If $g, h \in \widetilde{G}$ then

$$\pi'(\Phi(gh)) = \pi(g)\pi(h) = \pi'(\Phi(g)\Phi(h)),$$

so $(g,h) \mapsto \Phi(gh)$ and $(g,h) \mapsto \Phi(g)\Phi(h)$ are both lifts of $(g,h) \mapsto \pi(gh)$. Since the maps all agree at the point (e,e), we have $\Phi(gh) = \Phi(g)\Phi(h)$. □

Theorem 100. *[Problem 7-6] Suppose G is a Lie group and U is any neighborhood of the identity. There exists a neighborhood V of the identity such that $V \subseteq U$ and $gh^{-1} \in U$ whenever $g, h \in V$.*

Proof. Define $f(g,h) = gh^{-1}$ and let $W = f^{-1}(U)$. Since $(e,e) \in W$, there are neighborhoods W_1, W_2 of e such that $(e,e) \in W_1 \times W_2 \subseteq W$. Then $V = W_1 \cap W_2$ is the desired neighborhood of the identity. □

Theorem 101. *[Problem 7-7] Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

Proof. The subgroup H generated by G_0 is a connected open subgroup of G, so $H \subseteq G_0$ since G_0 is a connected component. Therefore $H = G_0$. For each $g \in G$, define the conjugation map $C_g : G \to G$ by $h \mapsto ghg^{-1}$. This map is a Lie group isomorphism, so $C_g(G_0)$ is a connected open subgroup of G and again we have $C_g(G_0) = G_0$. Every left coset gG_0 is the image of G_0 under the diffeomorphism L_g, so gG_0 is a connected open subset of G. But G is the union of all such cosets, so the connected components of G are precisely the cosets of G_0. $\square$

Theorem 102. *[Problem 7-8] Suppose a connected topological group G acts continuously on a discrete space K. Then the action is trivial.*

Proof. Let $\theta : G \times K \to K$ be the group action. For any $x \in K$ the set $\theta(G \times \{x\})$ is connected and therefore consists of one point. But $\theta(e, x) = x$, so $\theta(g, x) = x$ for every $g \in G$. $\square$

Theorem 103. *[Problem 7-9,7-10] The formula*

$$A \cdot [x] = [Ax]$$

defines a smooth, transitive left action of $\mathrm{GL}(n+1, \mathbb{R})$ *on* $\mathbb{RP}^n$*. The same formula defines a smooth, transitive left action of* $\mathrm{GL}(n+1, \mathbb{C})$ *on* $\mathbb{CP}^n$*.*

Proof. The smooth map $\widetilde{\theta} : \mathrm{GL}(n+1, \mathbb{R}) \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ given by $(A, x) \mapsto [Ax]$ is constant on each line in $\mathbb{R}^{n+1}$ since A is linear, so it descends to a smooth map $\theta : \mathrm{GL}(n+1, \mathbb{R}) \times \mathbb{RP}^n \to \mathbb{RP}^n$ given by $(A, [x]) \mapsto [Ax]$. It is easy to see that θ is a group action. Furthermore, for any two points $v, w \in \mathbb{R}^{n+1}$ there is an invertible linear map that takes v to w, so θ is transitive. $\square$

Example 104. [Problem 7-11] Considering $\mathbb{S}^{2n+1}$ as the unit sphere in $\mathbb{C}^{n+1}$, define an action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$, called the **Hopf action**, by

$$z \cdot (w^1, \ldots, w^{n+1}) = (zw^1, \ldots, zw^{n+1}).$$

This action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$ whose union is $\mathbb{S}^{2n+1}$.

Theorem 105. *[Problem 7-12] Every Lie group homomorphism has constant rank.*

Proof. If $F : G \to H$ is a Lie group homomorphism, then we can apply Theorem 7.25 with the left translation action on G and the action $g \cdot h = F(g)h$ on H. $\square$

Theorem 106. *[Problem 7-13,7-14] For each $n \geq 1$, $\mathrm{U}(n)$ is a properly embedded n^2-dimensional Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ is a properly embedded (n^2-1)-dimensional Lie subgroup of $\mathrm{U}(n)$.*

Proof. We can use the argument in Example 7.27, replacing the transpose operator with the conjugate transpose operator. □

Theorem 107. *[Problem 7-15] $\mathrm{SO}(2)$, $\mathrm{U}(1)$, and $\mathbb{S}^1$ are all isomorphic as Lie groups.*

Proof. $\mathrm{U}(1)$ is obviously identical to $\mathbb{S}^1$. We also have a Lie group isomorphism $f : \mathbb{S}^1 \to \mathrm{SO}(2)$ given by

$$f(z) = \begin{bmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta = \arg(z)$. □

Theorem 108. *[Problem 7-16] $\mathrm{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$.*

Proof. Identifying $\mathbb{S}^3$ with the subspace $\left\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\right\}$, we have a diffeomorphism

$$(z, w) \mapsto \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}.$$

□

Example 109. [Problem 7-17] Determine which of the following Lie groups are compact:

$$\mathrm{GL}(n, \mathbb{R}),\ \mathrm{SL}(n, \mathbb{R}),\ \mathrm{GL}(n, \mathbb{C}),\ \mathrm{SL}(n, \mathbb{C}),\ \mathrm{U}(n),\ \mathrm{SU}(n).$$

Clearly $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are not compact. $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ are not compact, since the operator norm of

$$\begin{bmatrix} r & & \\ & r^{-1} & \\ & & I \end{bmatrix} \in \mathrm{SL}(n, \mathbb{R})$$

is unbounded as $r \to \infty$. However, $\mathrm{U}(n)$ is compact since every unitary matrix has operator norm 1, and $\mathrm{SU}(n)$ is compact because it is a properly embedded subgroup of $\mathrm{U}(n)$.

Theorem 110. *[Problem 7-18] Suppose G is a Lie group, and $N, H \subseteq G$ are Lie subgroups such that N is normal, $N \cap H = \{e\}$, and $NH = G$. Then the map $(n, h) \mapsto nh$ is a Lie group isomorphism between $N \rtimes_\theta H$ and G, where $\theta : H \times N \to N$ is the action by conjugation: $\theta_h(n) = hnh^{-1}$. Furthermore, N and H are closed in G.*

Proof. Let φ be the map $(n,h)\mapsto nh$. If $nh=e$ then $n=h^{-1}\in N\cap H$, so $n=h=e$. Also, φ is surjective since $NH=G$. Therefore φ is a bijection. We have

$$\begin{aligned}\varphi((n,h)(n',h')) &= \varphi(nhn'h^{-1},hh')\\ &= nhn'h^{-1}hh'\\ &= nhn'h'\\ &= \varphi(n,h)\varphi(n',h'),\end{aligned}$$

so φ is an isomorphism. Since φ is smooth, it is a Lie group isomorphism. □

Theorem 111. *[Problem 7-19] Suppose G, N, and H are Lie groups. Then G is isomorphic to a semidirect product $N\rtimes H$ if and only if there are Lie group homomorphisms $\varphi: G\to H$ and $\psi: H\to G$ such that $\varphi\circ\psi=\mathrm{Id}_H$ and $\ker\varphi\cong N$.*

Proof. Suppose there is an isomorphism $f: G\to N\rtimes_\theta H$. Let $\pi: N\rtimes_\theta H\to H$ be the projection $(n,h)\mapsto h$ and let $\iota: H\to N\rtimes_\theta H$ be the injection $h\mapsto(e,h)$. Take $\varphi=\pi\circ f$ and $\psi=\iota$; then $\varphi\circ\psi=\mathrm{Id}_H$ and $\ker\varphi\cong N$. Conversely, suppose the homomorphisms φ and ψ exist. Let $N=\ker\varphi$ and $H_1=\psi(H)$. Then N is normal in G and $N\cap H_1=\{e\}$, since $x=\psi(h)$ and $x\in\ker\varphi$ implies that $h=\varphi(\psi(h))=\varphi(x)=e$. For any $g\in G$ we have $\varphi(\psi(\varphi(g)))=\varphi(g)$, so $g=n\psi(\varphi(g))$ for some $n\in\ker\varphi$. Therefore $NH_1=G$. By Theorem 110, G is a semidirect product. □

Example 112. [Problem 7-20] Prove that the following Lie groups are isomorphic to semidirect products as shown.

(1) $\mathrm{O}(n)\cong\mathrm{SO}(n)\rtimes\mathrm{O}(1)$.
(2) $\mathrm{U}(n)\cong\mathrm{SU}(n)\rtimes\mathrm{U}(1)$.
(3) $\mathrm{GL}(n,\mathbb{R})\cong\mathrm{SL}(n,\mathbb{R})\rtimes\mathbb{R}^*$.
(4) $\mathrm{GL}(n,\mathbb{C})\cong\mathrm{SL}(n,\mathbb{C})\rtimes\mathbb{C}^*$.

For (1) to (4), define φ by $M\mapsto\det(M)$ and ψ by

$$z\mapsto\begin{bmatrix}z & \\ & I_{n-1}\end{bmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, and apply Theorem 111.

Theorem 113. *[Problem 7-23] Let $\mathbb{H}^*$ be the Lie group of nonzero quaternions, and let $\mathcal{S}\subseteq\mathbb{H}^*$ be the set of unit quaternions. Then $\mathcal{S}$ is a properly embedded Lie subgroup of $\mathbb{H}^*$, isomorphic to* $\mathrm{SU}(2)$.

Proof. The norm $|\cdot|:\mathbb{H}^*\to\mathbb{R}^*$ is a smooth homomorphism, since

$$|(a,b)|=\sqrt{(a,b)(\bar{a},-b)}=\sqrt{|a|^2+|b|^2}$$

and $|pq| = |p|\,|q|$. Therefore $\ker|\cdot| = \mathcal{S}$ is a properly embedded Lie subgroup of $\mathbb{H}^*$. Furthermore, we have a Lie group isomorphism

$$(a,b) \mapsto \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$$

into SU(2). □

Chapter 8. Vector Fields

Theorem 114. *[Exercise 8.29] For $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$, we have*

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Proof. We compute

$$\begin{aligned} [fX, gY]h &= fX(gYh) - gY(fXh) \\ &= gfXYh + YhfXg - fgYXh - XhgYf \\ &= fg[X,Y]h + (fXg)Yh - (gYf)Xh. \end{aligned}$$

□

Theorem 115. *[Exercise 8.34] Let $A : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then* $\ker A$ *and* $\operatorname{im} A$ *are Lie subalgebras.*

Proof. It suffices to show that $\ker A$ and $\operatorname{im} A$ are closed under brackets. If $X, Y \in \ker A$ then $A[X,Y] = [AX, AY] = 0$, so $[X,Y] \in \ker A$. Similarly, if $X, Y \in \operatorname{im} A$ then $X = AX'$ and $Y = AY'$ for some $X', Y' \in \mathfrak{g}$, so $A[X', Y'] = [AX', AY'] = [X,Y]$ and $[X,Y] \in \operatorname{im} A$. □

Theorem 116. *[Exercise 8.35] Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are finite-dimensional Lie algebras and $A : \mathfrak{g} \to \mathfrak{h}$ is a linear map. Then A is a Lie algebra homomorphism if and only if $A[E_i, E_j] = [AE_i, AE_j]$ for some basis $(E_1, \ldots, E_n)$ of $\mathfrak{g}$.*

Proof. One direction is obvious. Suppose that $(E_1, \ldots, E_n)$ is a basis of $\mathfrak{g}$ and $A[E_i, E_j] = [AE_i, AE_j]$. Let $X, Y \in \mathfrak{g}$ and write $X = \sum_{i=1}^n x_i E_i$ and $Y = \sum_{i=1}^n y_i E_i$; then

$$\begin{aligned} A[X,Y] &= A\left[\sum_{i=1}^n x_i E_i, \sum_{j=1}^n y_j E_j\right] \\ &= \sum_{i=1}^n x_i \sum_{j=1}^n y_j A\left[E_i, E_j\right] \\ &= \sum_{i=1}^n x_i \sum_{j=1}^n y_j \left[AE_i, AE_j\right] \end{aligned}$$

$$= \left[A \sum_{i=1}^{n} x_i E_i, A \sum_{j=1}^{n} y_j E_j \right]$$
$$= [AX, AY].$$

□

Theorem 117. *[Exercise 8.43] If V is any finite-dimensional real vector space, the composition of canonical isomorphisms*

$$\mathrm{Lie}(\mathrm{GL}(V)) \to T_{\mathrm{Id}} \mathrm{GL}(V) \to \mathfrak{gl}(V)$$

yields a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{GL}(V))$ *and* $\mathfrak{gl}(V)$.

Proof. Choose a basis B for V and let $\varphi : \mathrm{GL}(V) \to \mathrm{GL}(n, \mathbb{R})$ be the associated Lie group isomorphism. It is also a Lie algebra isomorphism $\mathfrak{gl}(V) \to \mathfrak{gl}(n, \mathbb{R})$ since it is linear. By Corollary 8.31 there is a Lie algebra isomorphism $\rho : \mathrm{Lie}(\mathrm{GL}(V)) \to \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R}))$ induced by the diffeomorphism φ, and by Proposition 8.41 there is a Lie algebra isomorphism $\psi : \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) \to \mathfrak{gl}(n, \mathbb{R})$. Then $\varphi^{-1} \circ \psi \circ \rho$ is the desired Lie algebra isomorphism. □

Theorem 118. *[Problem 8-1] Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\operatorname{supp} \widetilde{X} \subseteq U$.*

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth vector field $\widetilde{X}_p$ on W_p that agrees with X on $W_p \cap A$. Replacing W_p by $W_p \cap U$ we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{X}_p$ is smooth on W_p by Proposition 8.8, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \setminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\widetilde{X} : M \to TM$ by

$$\widetilde{X}_x = \sum_{p \in A} \psi_p(x) \widetilde{X}_p|_x.$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{X}_p|_x = X_x$ for each p such that $\psi_p(x) \neq 0$, so

$$\widetilde{X}_x = \sum_{p \in A} \psi_p(x) X_x = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) X_x = X_x,$$

so $\widetilde{X}$ is indeed an extension of X. It follows from Lemma 1.13(b) that

$$\operatorname{supp} \widetilde{X} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup_{p \in A} \operatorname{supp} \psi_p \subseteq U.$$

□

Theorem 119. *[Problem 8-2] Let c be a real number, and let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a smooth function that is* ***positively homogeneous of degree*** *c, meaning that $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Then $Vf = cf$, where V is the Euler vector field defined in Example 8.3.*

Proof. We want to compute

$$(Vf)(x) = V_x f = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x).$$

Differentiating $\lambda^c f(x) = f(\lambda x)$ with respect to λ on $(0, \infty)$, we have

$$c\lambda^{c-1} f(x) = Df(\lambda x)(x) = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(\lambda x).$$

Replacing x with $\lambda^{-1} x$ gives

$$\begin{aligned} c\lambda^{c-1} f(\lambda^{-1} x) &= \lambda^{-1} \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x) \\ \Rightarrow c\lambda^{c-1} \lambda^{-c} f(x) &= \lambda^{-1} \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x) \\ \Rightarrow cf(x) &= \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x) = (Vf)(x). \end{aligned}$$

□

Theorem 120. *[Problem 8-3] Let M be a nonempty positive-dimensional smooth manifold with or without boundary. Then $\mathfrak{X}(M)$ is infinite-dimensional.*

Proof. Let $n = \dim M$. If $n \geq 2$ then the result is clear from Proposition 8.7 since there are infinitely many lines in $\mathbb{R}^2$, i.e. $\mathbb{RP}^1$ is infinite. Suppose $n = 1$ and choose any diffeomorphism $\varphi : (-\varepsilon, 1+\varepsilon) \to U$ where $\varepsilon > 0$ and U is open in M. For each $m = 1, 2, \ldots$, define a vector field X_m along $\{1/k : k = 1, \ldots, m\}$ by setting

$$X_m|_{1/k} = \begin{cases} 0, & k < m, \\ \frac{d}{dt}\big|_{1/k}, & k = m. \end{cases}$$

For each m, we have an extension $\widetilde{X}_m$ defined on $(-\varepsilon, 1+\varepsilon)$ by Lemma 8.6. We will show that $\{\widetilde{X}_m : m = 1, 2, \ldots\}$ is linearly independent by induction. Since $\widetilde{X}_m$ is nonzero, the set $\{\widetilde{X}_{m_0}\}$ is linearly independent for any m_0. Now suppose

$$a_1 \widetilde{X}_{m_1} + \cdots + a_r \widetilde{X}_{m_r} = 0$$

with $m_1 < \cdots < m_r$. Then

$$a_1 \widetilde{X}_{m_1}|_{1/m_1} + \cdots + a_r \widetilde{X}_{m_r}|_{1/m_1} = a_1 \widetilde{X}_{m_1}|_{1/m_1} = 0,$$

so $a_1 = 0$ since $\widetilde{X}_{m_1}|_{1/m_1} \neq 0$, and $a_2 = \cdots = a_r = 0$ by induction. Therefore $\{\varphi_* \widetilde{X}_m : m = 1, 2, \ldots\}$ is a set of linearly independent vector fields defined on U. We can extend these vector fields to M by restricting each $\varphi_* \widetilde{X}_m$ to the closed set $\varphi([0,1])$ and applying Lemma 8.6. This proves that $\mathfrak{X}(M)$ is infinite-dimensional. $\square$

Theorem 121. *[Problem 8-4] Let M be a smooth manifold with boundary. There exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.*

Proof. Let $n = \dim M$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a collection of smooth boundary charts whose domains cover ∂M. For each α, let X_α be the nth coordinate vector field. Let $\{\psi_\alpha\}$ be a partition of unity of $U = \bigcup_\alpha U_\alpha$ subordinate to $\{U_\alpha\}$; then $X = \sum_\alpha \psi_\alpha X_\alpha$ is a smooth vector field defined on U. For any boundary defining function $f : M \to [0, \infty)$ we have

$$X_p f = \sum_\alpha \psi_\alpha X_\alpha|_p f > 0$$

since each $X_\alpha|_p$ is inward-pointing, so X_p is always inward-pointing. By restricting X to ∂M and applying Lemma 8.6, we have a global smooth vector field $\widetilde{X}$ whose restriction to ∂M is everywhere inward-pointing. Also, the restriction of $-\widetilde{X}$ to ∂M is everywhere outward-pointing. $\square$

Theorem 122. *[Problem 8-5] Let M be a smooth n-manifold with or without boundary.*

(1) *If $(X_1, \ldots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an open subset $U \subseteq M$, with $1 \leq k < n$, then for each $p \in U$ there exist smooth*

vector fields $X_{k+1},\ldots,X_n$ in a neighborhood V of p such that $(X_1,\ldots,X_n)$ is a smooth local frame for M on $U\cap V$.

(2) *If $(v_1,\ldots,v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p\in M$, with $1\le k\le n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i|_p=v_i$ for $i=1,\ldots,k$.*

(3) *If $(X_1,\ldots,X_n)$ is a linearly independent n-tuple of smooth vector fields along a closed subset $A\subseteq M$, then there exists a smooth local frame $(\widetilde{X}_1,\ldots,\widetilde{X}_n)$ on some neighborhood of A such that $\widetilde{X}_i|_A=X_i$ for $i=1,\ldots,n$.*

Proof. See Theorem 162. □

Theorem 123. *[Problem 8-6] Let $\mathbb{H}$ be the algebra of quaternions and let $\mathcal{S}\subseteq\mathbb{H}$ be the group of unit quaternions.*

(1) *If $p\in\mathbb{H}$ is imaginary, then qp is tangent to $\mathcal{S}$ at each $q\in\mathcal{S}$.*

(2) *Define vector fields X_1,X_2,X_3 on $\mathbb{H}$ by*

$$X_1|_q=qi,\quad X_2|_q=qj,\quad X_3|_q=qk.$$

These vector fields restrict to a smooth left-invariant global frame on $\mathcal{S}$.

(3) *Under the isomorphism $(x^1,x^2,x^3,x^4)\leftrightarrow x^1 1+x^2 i+x^3 j+x^4 k$ between $\mathbb{R}^4$ and $\mathbb{H}$, these vector fields have the following coordinate representations:*

$$\begin{aligned}X_1&=-x^2\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2}+x^4\frac{\partial}{\partial x^3}-x^3\frac{\partial}{\partial x^4},\\X_2&=-x^3\frac{\partial}{\partial x^1}-x^4\frac{\partial}{\partial x^2}+x^1\frac{\partial}{\partial x^3}+x^2\frac{\partial}{\partial x^4},\\X_3&=-x^4\frac{\partial}{\partial x^1}+x^3\frac{\partial}{\partial x^2}-x^2\frac{\partial}{\partial x^3}+x^1\frac{\partial}{\partial x^4}.\end{aligned}$$

Proof. $\mathcal{S}$ is a level set of the norm squared $|\cdot|^2:\mathbb{H}^*\to(0,\infty)$, which is given by

$$|(a+bi,c+di)|^2=a^2+b^2+c^2+d^2$$

and has derivative

$$\begin{bmatrix}2a & 2b & 2c & 2d\end{bmatrix}.$$

If $q=(a,b,c,d)$ then by Theorem 65, $T_q\mathcal{S}$ is the kernel of this matrix. If $p\in\mathbb{H}$ is imaginary then p has the form $(ei,f+gi)$, so

$$qp=(-be-cf-dg+(ae+cg-df)i,af-bg+de+(ag+bf-ce)i)$$

and

$$a(-be-cf-dg)+b(ae+cg-df)+c(af-bg+de)+d(ag+bf-ce)=0.$$

Therefore $qp \in T_q\mathcal{S}$. This proves (1). For (2), let $g = (a, b, c, d)$ and $g' = (e + fi, g + hi)$. We compute

$$DL_g(g') = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

Then

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} -f \\ e \\ h \\ -g \end{bmatrix} = \begin{bmatrix} -(be + af - dg + ch) \\ ae - bf - cg - dh \\ de - cf + bg + ah \\ -(ce + df + ag - bh) \end{bmatrix},$$

which shows that $d(L_g)_{g'}(X_1|_{g'}) = X_1|_{gg'}$, i.e. X_1 is left-invariant. Similar calculations show that X_2 and X_3 are left-invariant. □

Example 124. [Problem 8-9] Show by finding a counterexample that Proposition 8.19 is false if we replace the assumption that F is a diffeomorphism by the weaker assumption that it is smooth and bijective.

Consider the smooth bijection $\omega : [0, 1) \to \mathbb{S}^1$ given by $s \mapsto e^{2\pi i s}$ and consider the smooth vector field X given by $x \mapsto (1 - 2x)\, d/dt|_x$. There is no way of defining an F-related smooth vector field Y on $\mathbb{S}^1$, since the sign of Y_1 is ambiguous.

Example 125. [Problem 8-10] Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F : M \to M$ be the map $F(x, y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}.$$

The inverse

$$F^{-1}(u, v) = \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right)$$

is smooth, so F is a diffeomorphism. We compute

$$DF(x, y) = \begin{bmatrix} y & x \\ -y/x^2 & 1/x \end{bmatrix}$$

so that

$$DF(F^{-1}(u, v)) = \begin{bmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{bmatrix}.$$

Therefore the coordinates of F_*X are given by

$$\begin{bmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{bmatrix} \begin{bmatrix} \sqrt{u/v} \\ \sqrt{uv} \end{bmatrix} = \begin{bmatrix} 2u \\ 0 \end{bmatrix},$$

and the coordinates of F_*Y are given by

$$\begin{bmatrix} \sqrt{uv} & \sqrt{u/v} \\ -v\sqrt{v/u} & \sqrt{v/u} \end{bmatrix} \begin{bmatrix} \sqrt{uv} \\ 0 \end{bmatrix} = \begin{bmatrix} uv \\ -v^2 \end{bmatrix}.$$

So

$$F_*X = 2u\frac{\partial}{\partial u}, \quad F_*Y = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}.$$

Example 126. [Problem 8-11] For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x, y) : x > 0\}$.

(1) $X = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$.
(2) $Y = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.
(3) $Z = (x^2 + y^2)\dfrac{\partial}{\partial x}$.

Let M be the right half-plane. We have a diffeomorphism $F : M \to (0, \infty) \times (-\pi, \pi)$ given by $F(x, y) = (\sqrt{x^2 + y^2}, \arctan(y/x))$, with inverse $F^{-1}(r, \theta) = (r\cos\theta, r\sin\theta)$. Then

$$DF(x, y) = \begin{bmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{bmatrix},$$

so

$$DF(F^{-1}(r, \theta)) = \begin{bmatrix} \cos\theta & \sin\theta \\ -r^{-1}\sin\theta & r^{-1}\cos\theta \end{bmatrix}.$$

We have

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -r^{-1}\sin\theta & r^{-1}\cos\theta \end{bmatrix} \begin{bmatrix} r\cos\theta & r\cos\theta & r^2 \\ r\sin\theta & -r\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} r & r\cos 2\theta & r^2\cos\theta \\ 0 & -\sin 2\theta & -r\sin\theta \end{bmatrix}.$$

Therefore

$$\begin{aligned} X &= r\frac{\partial}{\partial r}, \\ Y &= r\cos 2\theta\frac{\partial}{\partial r} - \sin 2\theta\frac{\partial}{\partial\theta}, \\ Z &= r^2\cos\theta\frac{\partial}{\partial r} - r\sin\theta\frac{\partial}{\partial\theta}. \end{aligned}$$

Example 127. [Problem 8-12] Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the smooth map $F(x, y) = [x, y, 1]$, and let $X \in \mathfrak{X}(\mathbb{R}^2)$ be defined by $X = x\partial/\partial y - y\partial/\partial x$. Prove that there is a vector field $Y \in \mathfrak{X}(\mathbb{RP}^2)$ that is F-related to X, and compute its coordinate representation in terms of each of the charts defined in Example 1.5. Let $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ be the quotient map and define $\widetilde{X} : \mathbb{R}^3 \setminus \{0\} \to T\mathbb{RP}^2$ by

$$\widetilde{X}_{(x,y,z)} = d\pi \left(x \left.\frac{\partial}{\partial y}\right|_{(x,y,z)} - y \left.\frac{\partial}{\partial x}\right|_{(x,y,z)} \right).$$

Since $\widetilde{X}$ is constant on the fibers of π, it descends to a vector field $Y : \mathbb{RP}^2 \to T\mathbb{RP}^2$. Let $\widetilde{U}_i \subseteq \mathbb{R}^3 \setminus \{0\}$, $U_i \subseteq \mathbb{RP}^2$ and $\varphi_i : U_i \to \mathbb{R}^n$ be as in Example 1.5:

$$\begin{aligned} \varphi_i[x^1, x^2, x^3] &= (x^1/x^i, x^2/x^i, x^3/x^i), \\ \varphi_i^{-1}(u^1, u^2, u^3) &= [u^1, \ldots, u^{i-1}, 1, u^{i+1}, \ldots, u^3]. \end{aligned}$$

The coordinate representations of $d\pi$ and Y with respect to φ_3 are given by

$$d\pi(u^1, u^2, u^3, v^1, v^2, v^3) = \left(\frac{u^1}{u^3}, \frac{u^2}{u^3}, 1, \frac{v^1}{u^3} - \frac{u^1 v^3}{(u^3)^2}, \frac{v^2}{u^3} - \frac{u^2 v^3}{(u^3)^2}, 0 \right)$$

and

$$\begin{aligned} Y(u^1, u^2, u^3) &= d\pi \left(u^1 \left.\frac{\partial}{\partial y}\right|_{(u^1,u^2,1)} - u^2 \left.\frac{\partial}{\partial x}\right|_{(u^1,u^2,1)} \right) \\ &= (u^1, u^2, 1, -u^2, u^1, 0). \end{aligned}$$

On this chart, the coordinate representation of dF is

$$dF(u^1, u^2, v^1, v^2) = (u^1, u^2, 1, v^1, v^2, 0),$$

so for all $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} dF_{(x,y)}(X_{(x,y)}) &= dF \left(x \left.\frac{\partial}{\partial y}\right|_{(x,y,z)} - y \left.\frac{\partial}{\partial x}\right|_{(x,y,z)} \right) \\ &= (x, y, 1, -y, x, 0) \\ &= Y(x, y, 1). \end{aligned}$$

The computations for φ_1 and φ_2 are similar. This shows that Y is F-related to X.

Theorem 128. *[Problem 8-13] There is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point.*

Proof. Let $N = (0, 0, 1)$ be the north pole and let $\sigma : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection given by

$$\begin{aligned} \sigma(x^1, x^2, x^3) &= (x^1, x^2)/(1 - x^3), \\ \sigma^{-1}(u^1, u^2) &= (2u^1, 2u^2, |u|^2 - 1)/(|u|^2 + 1). \end{aligned}$$

Let X be the 1st coordinate vector field on $\mathbb{R}^2$ and define a vector field Y on $\mathbb{S}^2$ by

$$Y(p) = \begin{cases} 0, & p = N, \\ ((\sigma^{-1})_* X)_p & p \neq N. \end{cases}$$

This vector field vanishes at exactly one point. Let $S = (0, 0, -1)$ be the south pole and let $\tau : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$ be the stereographic projection given by

$$\begin{aligned} \tau(x^1, x^2, x^3) &= (x^1, x^2)/(1 + x^3), \\ \tau^{-1}(u^1, u^2) &= (2u^1, 2u^2, 1 - |u|^2)/(|u|^2 + 1). \end{aligned}$$

Then

$$(\sigma \circ \tau^{-1})(u^1, u^2) = (2u^1, 2u^2)/(2\,|u|^2),$$

and the coordinate representation of Y for $(u^1, u^2) \neq (0, 0)$ with respect to τ is given by

$$\begin{aligned} Y(u^1, u^2) &= d(\sigma^{-1})_{(\sigma\circ\tau^{-1})(u^1,u^2)} X_{(\sigma\circ\tau^{-1})(u^1,u^2)} \\ &= d(\sigma^{-1})_{(\sigma\circ\tau^{-1})(u^1,u^2)} \left(\left.\frac{d}{dx}\right|_{(\sigma\circ\tau^{-1})(u^1,u^2)} \right) \\ &= \left(\tau^{-1}(u^1, u^2), \frac{2(1 - (u^1)^2 + (u^2)^2)}{(1 + |u|^2)^2}, -\frac{4u^1u^2}{(1 + |u|^2)^2}, -\frac{4u^1}{(1 + |u|^2)^2} \right), \end{aligned}$$

which shows that Y is smooth on $\mathbb{S}^2$. □

Theorem 129. *[Problem 8-14] Let M be a smooth manifold with or without boundary, let N be a smooth manifold, and let $f : M \to N$ be a smooth map. Define $F : M \to M \times N$ by $F(x) = (x, f(x))$. For every $X \in \mathfrak{X}(M)$, there is a smooth vector field on $M \times N$ that is F-related to X.*

Proof. Let $m = \dim M$ and $n = \dim N$. By Lemma 8.6, it suffices to show that there is a smooth vector field along $F(M)$ that is F-related to X. By Proposition 5.4, $F(M)$ is an embedded m-submanifold of $M \times N$. Let $p \in F(M)$, let (U, φ) be a slice chart for $F(M)$ containing p, let $\widetilde{p} = \varphi(p)$, and let $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^{m+n}$. By shrinking U, we may assume that $\widetilde{U}$ is an open cube containing $\widetilde{p}$, and that $\varphi(F(M) \cap U) = \{(a, b) \in \mathbb{R}^{m+n} : b = 0\}$. We identify $T_{\widetilde{p}}\widetilde{U}$ with $\widetilde{U} \times \mathbb{R}^{m+n}$. Define a vector field Y on $\widetilde{U}$ by assigning $Y_{(a,b)}$ the coefficients of $d\varphi(X_{\pi(\varphi^{-1}(a,0))})$, where $\pi : M \times N \to M$ is the canonical projection. Then Y is smooth, and $(\varphi^{-1})_* Y$ is a smooth vector field defined on U that agrees with X on $F(M) \cap U$. □

Theorem 130. *[Problem 8-15] Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold with or without boundary. Given $X \in \mathfrak{X}(S)$, there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Every such vector field extends to all of M if S is properly embedded.*

Proof. As in Theorem 82. $\square$

Example 131. [Problem 8-16] For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

(1) $X = y\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}$; $\quad Y = \dfrac{\partial}{\partial y}$.
(2) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$; $\quad Y = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$.
(3) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$; $\quad Y = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}$.

For (1), we have

$$\frac{\partial X^2}{\partial x} = -2y^2, \quad \frac{\partial X^2}{\partial y} = -4xy, \quad \frac{\partial X^3}{\partial y} = 1,$$

so

$$[X, Y] = 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

For (2), we have

$$\frac{\partial X^1}{\partial y} = -1, \quad \frac{\partial X^2}{\partial x} = 1, \quad \frac{\partial Y^2}{\partial z} = -1, \quad \frac{\partial Y^3}{\partial y} = -1,$$

so

$$[X, Y] = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

For (3), we have

$$\frac{\partial X^1}{\partial y} = -1, \quad \frac{\partial X^2}{\partial x} = 1, \quad \frac{\partial Y^1}{\partial y} = 1, \quad \frac{\partial Y^2}{\partial x} = 1,$$

so

$$[X, Y] = 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}.$$

Theorem 132. *[Problem 8-17] Let M and N be smooth manifolds. Given vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, we can define a vector field $X \oplus Y$ on $M \times N$ by*

$$(X \oplus Y)_{(p,q)} = (X_p, Y_q),$$

where we think of the right-hand side as an element of $T_pM \oplus T_qN$, which is naturally identified with $T_{p,q}(M \times N)$ as in Proposition 3.14. Then $X \oplus Y$ is smooth if X and Y are smooth, and $[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, X_2] \oplus [Y_1, Y_2]$.

Proof. The smoothness of $X \oplus Y$ follows easily from Proposition 8.1. If $f \in C^\infty(M \times N)$ then

$$[X_1 \oplus Y_1, X_2 \oplus Y_2]_{(p,q)} f = (X_1 \oplus Y_1)_{(p,q)}(X_2 \oplus Y_2)f - (X_2 \oplus Y_2)_{(p,q)}(X_1 \oplus Y_1)f$$

$$
\begin{aligned}
&= (X_1|_p X_2 f_q, Y_1|_q Y_2 f_p) - (X_2|_p X_1 f_q, Y_2|_q Y_1 f_p) \\
&= ([X_1, X_2] f_q, [Y_1, Y_2] f_p) \\
&= ([X_1, X_2] \oplus [Y_1, Y_2]) f,
\end{aligned}
$$

where $f_q \in C^\infty(M)$ is the map $p \mapsto f(p,q)$ and $f_p \in C^\infty(N)$ is the map $q \mapsto f(p,q)$. □

Theorem 133. *[Problem 8-18] Suppose $F : M \to N$ is a smooth submersion, where M and N are positive-dimensional smooth manifolds. Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, we say that X is a* ***lift of*** *Y if X and Y are F-related. A vector field $V \in \mathfrak{X}(M)$ is said to be* ***vertical*** *if V is everywhere tangent to the fibers of F (or, equivalently, if V is F-related to the zero vector field on N).*

(1) *If $\dim M = \dim N$, then every smooth vector field on N has a unique lift.*
(2) *If $\dim M \neq \dim N$, then every smooth vector field on N has a lift, but it is not unique.*
(3) *Assume in addition that F is surjective. Given $X \in \mathfrak{X}(M)$, X is a lift of a smooth vector field on N if and only if $dF_p(X_p) = dF_q(X_q)$ whenever $F(p) = F(q)$. If this is the case, then X is a lift of a* unique *smooth vector field.*

Proof. Let $m = \dim M$ and $n = \dim N$. If $m = n$ then F is a local diffeomorphism, and if $Y \in \mathfrak{X}(N)$ then we can define a smooth vector field $X \in \mathfrak{X}(M)$ by

$$X_p = (dF_p)^{-1}(Y_{F(p)}).$$

This is clearly the only vector field that satisfies $dF_p(X_p) = Y_{F(p)}$, and smoothness follows from Theorem 41 and the fact that dF is a local diffeomorphism. This proves (1). Now suppose $Y \in \mathfrak{X}(N)$ and $m \neq n$, i.e. $m > n$. Let $p \in M$, let $q = F(p)$, and choose smooth coordinates (x^i) centered at p and (y^i) centered at q in which F has the coordinate representation $\pi(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$. If ε is a sufficiently small number, the coordinate cube

$$C_p = \left\{x : \left|x^i\right| < \varepsilon \text{ for } i = 1, \ldots, m\right\}$$

is a neighborhood of p whose image under F is the cube

$$C_p' = \left\{y : \left|y^i\right| < \varepsilon \text{ for } i = 1, \ldots, n\right\}.$$

Define a smooth vector field X_p on C_p by setting $X_p^i(x) = Y^i(F(x))$ for $i = 1, \ldots, n$ and $X_p^i(x) = 0$ for $i = n+1, \ldots m$. Then $dF_x(X_p|_x) = Y_{F(x)}$ for all $x \in C_p$. Choose a partition of unity $\{\psi_p\}$ subordinate to the open cover $\{C_p\}$. Then

$$X = \sum_{p \in M} \psi_p X_p$$

is a smooth vector field on M (taking $\psi_p X_p = 0$ on $M \setminus \operatorname{supp}\psi_p$), and

$$\begin{aligned} dF_x(X_x) &= dF_x\left(\sum_{p\in M} \psi_p(x) X_p|_x\right) \\ &= \sum_{p\in M} \psi_p(x) dF_x(X_p|_x) \\ &= \sum_{p\in M} \psi_p(x) Y_{F(x)} \\ &= Y_{F(x)} \end{aligned}$$

for all $x \in M$. This proves (2). For (3), F is a quotient map. If X is a lift of some $Y \in \mathfrak{X}(N)$ then for all $p, q \in M$,

$$dF_p(X_p) = Y_{F(p)} = Y_{F(q)} = dF_q(X_q).$$

Conversely, if $dF_p(X_p) = dF_q(X_q)$ whenever $F(p) = F(q)$ then $dF \circ X$ is constant on the fibers of F, so it descends to a smooth vector field Y on N satisfying $dF \circ X = Y \circ F$, i.e. $dF_p(X_p) = Y_{F(p)}$ for all $p \in M$. □

Theorem 134. *[Problem 8-20] Let $A \subseteq \mathfrak{X}(\mathbb{R}^3)$ be the subspace spanned by $\{X, Y, Z\}$, where*

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Then A is a Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$ which is isomorphic to $\mathbb{R}^3$ with the cross product.

Proof. Since $\{X, Y, Z\}$ is linearly independent, it is a basis for A. Simple computations similar to those in Example 131 show that A is closed under brackets. Define an isomorphism $\varphi : A \to \mathbb{R}^3$ by setting $\varphi X = (1, 0, 0)$, $\varphi Y = (0, 1, 0)$, $\varphi Z = (0, 0, 1)$, and extending linearly. It is easy to check that φ is a Lie algebra isomorphism. For example,

$$\varphi[Z, X] = \varphi Y = [(0, 0, 1), (1, 0, 0)] = [\varphi Z, \varphi X].$$

□

Theorem 135. *[Problem 8-21] Up to isomorphism, there are exactly one 1-dimensional Lie algebra and two 2-dimensional Lie algebras. All three algebras are isomorphic to Lie subalgebras of $\mathfrak{gl}(2, \mathbb{R})$.*

Proof. Any 1-dimensional Lie algebra is abelian, and is isomorphic to the subalgebra $\{rI_2 : r \in \mathbb{R}\}$ of $\mathfrak{gl}(2, \mathbb{R})$. Let A be a 2-dimensional Lie algebra and let $\{x, y\}$ be a basis for A. If A is abelian then it is isomorphic to the subalgebra

$$\left\{ \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, r_1, r_2 \in \mathbb{R} \right\}$$

of $\mathfrak{gl}(2,\mathbb{R})$. Otherwise, we have $[x,y]=ax+by$ for some $a,b\in\mathbb{R}$, not both zero. Assume without loss of generality that $b\neq 0$. (If $b=0$, then we can swap x and y.) Define an homomorphism $\varphi : A\to\mathfrak{gl}(2,\mathbb{R})$ by setting

$$\varphi x=\begin{bmatrix}1 & 0\\ 0 & 1+b\end{bmatrix},\quad \varphi y=\begin{bmatrix}-ab^{-1} & 0\\ 1 & -ab^{-1}(1+b)\end{bmatrix}$$

and extending linearly. Then φ is an isomorphism onto its image, and it is easy to verify that $\varphi[x,y]=[\varphi x,\varphi y]$. □

Theorem 136. *[Problem 8-22] Let A be any algebra over $\mathbb{R}$. A* ***derivation of*** *A is a linear map $D : A\to A$ satisfying $D(xy)=(Dx)y+x(Dy)$ for all $x,y\in A$. If D_1 and D_2 are derivations of A, then $[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$ is also a derivation. The set of derivations of A is a Lie algebra with this bracket operation.*

Proof. Identical to Lemma 8.25. □

Theorem 137. *[Problem 8-23]*

(1) *Given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, the direct sum $\mathfrak{g}\oplus\mathfrak{h}$ is a Lie algebra with the bracket defined by*
$$[(X,Y),(X,Y')]=([X,X'],[Y,Y']).$$
(2) *Suppose G and H are Lie groups. Then* $\mathrm{Lie}(G\times H)$ *is isomorphic to* $\mathrm{Lie}(G)\oplus\mathrm{Lie}(H)$.

Proof. We have isomorphisms $\varepsilon : \mathrm{Lie}(G\times H)\to T_e(G\times H)$, $\varepsilon_1 : \mathrm{Lie}(G)\to T_e(G)$ and $\varepsilon_2 : \mathrm{Lie}(H)\to T_e(H)$. Thus we have an isomorphism

$$\varepsilon' : \mathrm{Lie}(G)\oplus\mathrm{Lie}(H)\to T_e(G)\oplus T_e(H)\cong T_e(G\times H),$$

and it remains to show that $\varepsilon^{-1}\circ\varepsilon'$ respects brackets. Let $(X_1,X_2),(Y_1,Y_2)\in\mathrm{Lie}(G)\oplus\mathrm{Lie}(H)$; then

$$\begin{aligned}(\varepsilon^{-1}\circ\varepsilon')[(X_1,X_2),(Y_1,Y_2)]_{(g,h)} &= ((\varepsilon^{-1}\circ\varepsilon')([X_1,Y_1],[X_2,Y_2]))_{(g,h)}\\ &= (\varepsilon^{-1}([X_1,Y_1]_e,[X_2,Y_2]_e))_{(g,h)}\\ &= d(L_{(g,h)})_{(e,e)}([X_1,Y_1]\oplus[X_2,Y_2])_{(e,e)}\\ &= d(L_{(g,h)})_{(e,e)}[X_1\oplus X_2,Y_1\oplus Y_2]_{(g,h)}\\ &= [X_1\oplus X_2,Y_1\oplus Y_2]_{(g,h)}\\ &= [(\varepsilon^{-1}\circ\varepsilon')(X_1,X_2),(\varepsilon^{-1}\circ\varepsilon')(Y_1,Y_2)]_{(g,h)}\end{aligned}$$

since $[X_1\oplus X_2,Y_1\oplus Y_2]$ is left-invariant (see Theorem 132). □

Theorem 138. *[Problem 8-24] Suppose G is a Lie group and $\mathfrak{g}$ is its Lie algebra. A vector field $X\in\mathfrak{X}(G)$ is said to be* ***right-invariant*** *if it is invariant under all right translations.*

(1) *The set* $\bar{\mathfrak{g}}$ *of right-invariant vector fields on* G *is a Lie subalgebra of* $\mathfrak{X}(G)$.
(2) *Let* $i : G \to G$ *denote the inversion map* $i(g) = g^{-1}$. *The pushforward* $i_* : \mathfrak{X}(G) \to \mathfrak{X}(G)$ *restricts to a Lie algebra isomorphism from* $\mathfrak{g}$ *to* $\bar{\mathfrak{g}}$.

Proof. Part (1) follows from Corollary 8.31. If $X \in \mathfrak{g}$ then for all $g, g' \in G$ we have

$$\begin{aligned} d(R_g)_{g'}((i_*X)_{g'}) &= (d(R_g)_{g'} \circ di_{i^{-1}(g')})(X_{i^{-1}(g')}) \\ &= d(R_g \circ i)_{(g')^{-1}}(X_{(g')^{-1}}) \\ &= d(i \circ L_{g^{-1}})_{(g')^{-1}}(X_{(g')^{-1}}) \\ &= di_{(g'g)^{-1}}(d(L_{g^{-1}})_{(g')^{-1}}X_{(g')^{-1}}) \\ &= di_{(g'g)^{-1}}(X_{(g'g)^{-1}}) \\ &= (i_*X)_{g'g}. \end{aligned}$$

Therefore $i_*(\mathfrak{g}) \subseteq \bar{\mathfrak{g}}$. A similar calculation shows that $i_*(\bar{\mathfrak{g}}) \subseteq \mathfrak{g}$, so $(i_*)^{-1} = i_*$ and i_* is an isomorphism. Corollary 8.31 shows that i_* is a Lie algebra isomorphism. □

Theorem 139. *[Problem 8-25] If* G *is an abelian Lie group, then* $\mathrm{Lie}(G)$ *is abelian.*

Proof. If G is abelian then the inversion map $i : G \to G$ is a Lie group isomorphism. The induced Lie algebra isomorphism is given by

$$\begin{aligned} (i_*X)_g &= d(L_g)_e(di_e(X_e)) \\ &= d(L_g)_e(-X_e) \\ &= -d(L_g)_eX_e \\ &= -X_g, \end{aligned}$$

so $i_*X = -X$ for all $X \in \mathrm{Lie}(G)$. Then

$$[X, Y] = [-X, -Y] = [i_*X, i_*Y] = i_*[X, Y] = -[X, Y],$$

so $[X, Y] = 0$ for all $X, Y \in \mathrm{Lie}(G)$. This proves that $\mathrm{Lie}(G)$ is abelian. □

Theorem 140. *[Problem 8-26] Suppose* $F : G \to H$ *is a Lie group homomorphism. The kernel of* $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *is the Lie algebra of* $\ker F$ *(under the identification of the Lie algebra of a subgroup with a Lie subalgebra as in Theorem 8.46).*

Proof. Consider ι_*X for some $X \in \ker F$, where $\iota : \ker F \hookrightarrow G$ is the inclusion map. Then $F_*\iota_*X = (F \circ \iota)_*X = 0$ since $F \circ \iota = 0$. This shows that $\iota_*(\mathrm{Lie}(\ker F)) \subseteq \ker F_*$. Conversely, suppose $F_*Y = 0$ for some $Y \in \mathrm{Lie}(G)$. By definition we have $dF_e(Y_e) = 0$, so $Y_e \in \ker(dF_e) = T_e(\ker F)$ by Theorem 65. By Theorem 8.46, this shows that $\ker F_* \subseteq \iota_*(\mathrm{Lie}(\ker F))$. □

Theorem 141. *[Problem 8-27] Let G and H be Lie groups, and suppose $F : G \to H$ is a Lie group homomorphism that is also a local diffeomorphism. The induced homomorphism $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is an isomorphism of Lie algebras.*

Proof. Since F is a local diffeomorphism, dF_e is bijective. If $F_*X = 0$ then $dF_e(X_e) = 0$, so $X_e = 0$ and $X = 0$. Suppose $Y \in \mathrm{Lie}(H)$. Let $\varepsilon : \mathrm{Lie}(G) \to T_eG$ be the canonical isomorphism. Then

$$\begin{aligned}(F_*\varepsilon^{-1}(dF_e)^{-1}Y_e)_e &= dF_e((\varepsilon^{-1}(dF_e)^{-1}Y_e)_e) \\ &= dF_e((dF_e)^{-1}Y_e) \\ &= Y_e,\end{aligned}$$

so $F_*(\varepsilon^{-1}(dF_e)^{-1}Y_e) = Y$. This proves that F_* is a bijection. □

Theorem 142. *[Problem 8-28] Considering* $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ *as a Lie group homomorphism, its induced Lie algebra homomorphism is* $\mathrm{tr} : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$.

Proof. This follows immediately from Theorem 98. □

Example 143. [Problem 8-29] Theorem 8.46 implies that the Lie algebra of any Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ is canonically isomorphic to a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$, with a similar statement for Lie subgroups of $\mathrm{GL}(n,\mathbb{C})$. Under this isomorphism, show that

$$\begin{aligned}\mathrm{Lie}(\mathrm{SL}(n,\mathbb{R})) &\cong \mathfrak{sl}(n,\mathbb{R}), \\ \mathrm{Lie}(\mathrm{SO}(n)) &\cong \mathfrak{o}(n), \\ \mathrm{Lie}(\mathrm{SL}(n,\mathbb{C})) &\cong \mathfrak{sl}(n,\mathbb{C}), \\ \mathrm{Lie}(\mathrm{U}(n)) &\cong \mathfrak{u}(n), \\ \mathrm{Lie}(\mathrm{SU}(n)) &\cong \mathfrak{su}(n),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{R}) &= \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}\, A = 0\}, \\ \mathfrak{o}(n) &= \{A \in \mathfrak{gl}(n,\mathbb{R}) : A^T + A = 0\}, \\ \mathfrak{sl}(n,\mathbb{C}) &= \{A \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}\, A = 0\}, \\ \mathfrak{u}(n) &= \{A \in \mathfrak{gl}(n,\mathbb{C}) : A^* + A = 0\}, \\ \mathfrak{su}(n) &= \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C}).\end{aligned}$$

We have $\mathrm{SL}(n,\mathbb{R}) = \ker\det$, so $\mathrm{Lie}(\mathrm{SL}(n,\mathbb{R})) \cong \ker\det_* = \ker\mathrm{tr} = \mathfrak{sl}(n,\mathbb{R})$. The same computation holds for $\mathrm{SL}(n,\mathbb{C})$. The others follow by considering the kernel of the map $A \mapsto A^TA$ (or $A \mapsto A^*A$ in the complex case).

Example 144. [Problem 8-30] Show by giving an explicit isomorphism that $\mathfrak{su}(2)$ and $\mathfrak{o}(3)$ are isomorphic Lie algebras, and that both are isomorphic to $\mathbb{R}^3$ with the cross product.

We have an isomorphism from $\mathbb{R}^3$ to $\mathfrak{o}(3)$ given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}.$$

We also have an isomorphism from $\mathbb{R}^3$ to $\mathfrak{su}(2)$ given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix} ai & b+ci \\ -b+ci & -ai \end{bmatrix}.$$

Theorem 145. *[Problem 8-31] Let $\mathfrak{g}$ be a Lie algebra. A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called an* ***ideal in*** *$\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.*

(1) *If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then the quotient space $\mathfrak{g}/\mathfrak{h}$ has a unique Lie algebra structure such that the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a Lie algebra homomorphism.*
(2) *A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if and only if it is the kernel of a Lie algebra homomorphism.*

Proof. Define a bracket on $\mathfrak{g}/\mathfrak{h}$ by $[X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h}$. If $X' + \mathfrak{h} = X + \mathfrak{h}$ and $Y' + \mathfrak{h} = Y + \mathfrak{h}$ then

$$[X', Y'] = [X' + X - X', Y'] = [X, Y' + Y - Y'] = [X, Y]$$

since $X - X' \in \mathfrak{h}$ and $Y - Y' \in \mathfrak{h}$, which shows that the bracket is well-defined. It is clearly the unique bracket making π a Lie algebra homomorphism. If $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal then it is the kernel of the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Conversely, if f is a Lie algebra homomorphism then $\ker f$ is an ideal since $X \in \ker f$ implies that

$$f[X, Y] = [fX, fY] = 0.$$

□

Chapter 9. Integral Curves and Flows

Theorem 146. *[Exercise 9.37] Suppose $v \in \mathbb{R}^n$ and W is a smooth vector field on an open subset of $\mathbb{R}^n$. The directional derivative*

$$D_v W(p) = \left.\frac{d}{dt}\right|_{t=0} W_{p+tv}$$

is equal to $(\mathcal{L}_V W)_p$, where V is the vector field $V = v^i \partial/\partial x^i$ with constant coefficients in standard coordinates.

Proof. The flow of V is given by $\theta(t,x) = x + tv$. Therefore

$$\begin{aligned}(\mathcal{L}_V W)_p &= \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) \\ &= \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{p+tv}(W_{p+tv}) \\ &= \frac{d}{dt}\bigg|_{t=0} W_{p+tv}\end{aligned}$$

since $\theta_{-t}(x) = x - tv$ is a translation of x. □

Theorem 147. *[Problem 9-1] Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.*

(1) *We say that γ is* ***periodic*** *if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Exactly one of the following holds:*
  (a) *γ is constant.*
  (b) *γ is injective.*
  (c) *γ is periodic and nonconstant.*

(2) *If γ is periodic and nonconstant, then there exists a unique positive number T (called the* ***period of*** *γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.*

(3) *The image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, or $\mathbb{R}^0$.*

Proof. If γ is constant then (b) and (c) cannot hold, so we assume that γ is nonconstant. If γ is not injective then $\gamma(t_0) = \gamma(t_1)$ for some points $t_0 < t_1$, and γ is defined on at least $(t_0 - \varepsilon, t_1 + \varepsilon)$ for some $\varepsilon > 0$. Let $T = t_1 - t_0$. Then γ and $t \mapsto \gamma(t_1 + t)$ are both integral curves of X starting at $\gamma(t_0)$, so Theorem 9.12 shows that γ must be defined on at least $(t_0 - \varepsilon, t_1 + T + \varepsilon)$. By induction, $t_0 + kT$ is in the domain of γ for all $k \in \mathbb{Z}$, so γ is defined on $\mathbb{R}$ and has period T. This proves (1). For (2), let A be the set of all $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$, which is nonempty since γ is periodic. If we can show that A is closed, then $T_0 = \inf A \in A$ is the period of γ. But if $T \notin A$ then $\gamma(t+T) \neq \gamma(t)$ for some $t \in \mathbb{R}$, so

$$\gamma^{-1}(M \setminus \{\gamma(t)\}) - t = \{s - t : \gamma(s) \neq \gamma(t)\}$$

is a neighborhood of T contained in $\mathbb{R} \setminus A$. For (3), if γ is constant then the image of γ is diffeomorphic to $\mathbb{R}^0$. Otherwise, Proposition 9.21 shows that γ is a smooth immersion. If γ is injective then Proposition 5.18 shows that the image of γ is diffeomorphic to $\mathbb{R}$, and if γ is periodic then it descends to a smooth injective immersion from $\mathbb{S}^1$ to M (using a suitable smooth covering of $\mathbb{S}^1$ by $\mathbb{R}$), which is a diffeomorphism onto its image by Proposition 5.18. □

Theorem 148. *[Problem 9-2] Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold, and V is a smooth vector field on M that is tangent to S.*

(1) *For any integral curve γ of V such that $\gamma(t_0) \in S$, there exists $\varepsilon > 0$ such that $\gamma((t_0 - \varepsilon, t_0 + \varepsilon)) \subseteq S$.*
(2) *If S is properly embedded, then every integral curve that intersects S is contained in S.*
(3) *(2) need not hold if S is not closed.*

Proof. Let $x = \gamma(t_0) \in S$. By Proposition 8.23, there is a smooth vector field $V|_S$ on S that is ι-related to V. Let $\alpha : I \to S$ be a maximal integral curve of $V|_S$ starting at x, where I contains $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Then

$$(\iota \circ \alpha)'(0) = d\iota_x(\alpha'(0)) = d\iota_x((V|_S)_x) = V_x,$$

which shows that $\iota \circ \alpha$ is an integral curve of V starting at x. By Theorem 9.12, γ is defined on $J = (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\gamma(t) = (\iota \circ \alpha)(t - t_0) \in S$ for $t \in J$. This proves (1). Now suppose that S is properly embedded and let γ be an integral curve that intersects S, i.e. $\gamma(t_0) \in S$ for some t_0. Let α be defined as above. By Theorem 9.12, γ is defined on at least $I + t_0 = (a, b)$ and $\gamma(t) \in S$ for all $t \in (a, b)$. Suppose γ can be extended to an open interval J larger than (a, b). Since S is closed, $\gamma(b) \in S$ by continuity. Then (1) shows that $\gamma((b - \varepsilon, b + \varepsilon)) \subseteq S$ for some $\varepsilon > 0$, which contradicts the maximality of α. This proves (2). For (3), let S be the open ball of radius 1 in $\mathbb{R}$ and let V be the 1st coordinate vector field on $\mathbb{R}$. Then $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\theta(t, x) = x + t$ is an integral curve of V that intersects S, but θ is not contained in S. □

Example 149. [Problem 9-3] Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

(1) $V = y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.
(2) $W = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$.
(3) $X = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.
(4) $Y = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$.

Let $\gamma(t) = (x(t), y(t))$ be a curve. We have the differential equations

$$\begin{aligned} x'(t) &= y(t), \quad y'(t) = 1, \\ x'(t) &= x(t), \quad y'(t) = 2y(t), \\ x'(t) &= x(t), \quad y'(t) = -y(t) \\ x'(t) &= y(t), \quad y'(t) = x(t). \end{aligned}$$

Therefore the flows (in this order) are

$$\begin{aligned}
\theta_t(x,y) &= \left(x + ty + \frac{1}{2}t^2, y + t\right),\\
\theta_t(x,y) &= (xe^t, ye^{2t}),\\
\theta_t(x,y) &= (xe^t, ye^{-t}),\\
\theta_t(x,y) &= \frac{1}{2}\left(e^t(x+y) + e^{-t}(x-y), e^t(x+y) - e^{-t}(x-y)\right).
\end{aligned}$$

Theorem 150. *[Problem 9-4] For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t,z) = e^{it}z$. The infinitesimal generator of θ is a smooth nonvanishing vector field on $\mathbb{S}^{2n-1}$.*

Proof. This is obvious. □

Theorem 151. *[Problem 9-5] Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. There exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.*

Proof. Let V be a nowhere vanishing smooth vector field on M. Since M is compact, Corollary 9.17 shows that there is a global flow $\theta : \mathbb{R} \times M \to M$ of V. By Theorem 9.22, each p has a neighborhood U_p in which V has the coordinate representation $\partial/\partial s^1$. Choosing a sufficiently small neighborhood V_p of p, there is some $T_p > 0$ such that $\theta_t(x) = x + (t, 0, \ldots, 0)$ in local coordinates for all $0 \leq t \leq T_p$ and $x \in V_p$. Since $\{V_p : p \in M\}$ is an open cover of M, there is a finite subcover $\{V_{p_1}, \ldots, V_{p_n}\}$. Let $T = \min\{T_{p_1}, \ldots, T_{p_n}\}$. Then θ_T has no fixed points, and the map $H : M \times I \to M$ given by $(x,t) \mapsto \theta(tT, x)$ is a homotopy from the identity to θ_T. □

Theorem 152. *[Problem 9-6] Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.*

Proof. Let θ be the flow of V. Let $\{b_n\}$ be an increasing sequence contained in $[t_0, b)$ and converging to b. Since $\gamma([t_0, b))$ is contained in a compact set $E \subseteq M$, there is a subsequence $\{\gamma(b_{n_k})\}$ of $\{\gamma(b_n)\}$ that converges to a point p in E. Choose some $\varepsilon > 0$ and a neighborhood U of p such that θ is defined on $(-2\varepsilon, 2\varepsilon) \times U$, and choose an integer m such that $b_m \in (b - \varepsilon, b)$ and $\gamma(b_m) \in U$. Define $\gamma_1 : [t_0, b + \varepsilon) \to M$ by

$$\gamma_1(t) = \begin{cases} \gamma(t), & t_0 \leq t < b, \\ \theta^{(\gamma(b_m))}(t - b_m) & b \leq t < b + \varepsilon. \end{cases}$$

For all $t \in (b_m, b)$ we have

$$\theta^{(\gamma(b_m))}(t - b_m) = \theta_{t - b_m}(\gamma(b_m)) = \gamma(t),$$

so γ_1 is smooth because γ and $t \mapsto \theta^{(\gamma(b_m))}(t - b_m)$ agree where they overlap. But γ_1 extends γ at b, which contradicts the maximality of γ. $\square$

Theorem 153. *[Problem 9-7] Let M be a connected smooth manifold. The group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$.*

Proof. First assume that $M = \mathbb{B}^n$. By applying Lemma 8.6 to the map $x \mapsto q - p$ defined on the line segment between p and q, we have a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$. Theorem 9.16 shows that θ_1 is a diffeomorphism on $\mathbb{B}^n$, which proves the result for $M = \mathbb{B}^n$. The general case follows from Theorem 29 and the fact that every point of M is contained in a coordinate ball. $\square$

Theorem 154. *[Problem 9-8] Let M be a smooth manifold and let $S \subseteq M$ be a compact embedded submanifold. Suppose $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. There exists $\varepsilon > 0$ such that the flow of V restricts to a smooth embedding $\Phi : (-\varepsilon, \varepsilon) \times S \to M$.*

Proof. Let $\theta : \mathcal{D} \to M$ be the flow of V and let $\Phi = \theta_{\mathcal{O}}$ where $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$. Let $\delta : S \to \mathbb{R}$ be as in Theorem 9.20. Since S is compact, δ attains a minimum α on S, and $\Phi : (-\alpha, \alpha) \times S \to M$ is a smooth immersion. Let $\varepsilon = \alpha/2$. Then $[-\varepsilon, \varepsilon] \times S$ is compact, so $\Phi : [-\varepsilon, \varepsilon] \times S \to M$ is a smooth embedding. Therefore $\Phi : (-\varepsilon, \varepsilon) \times S \to M$ is also a smooth embedding. $\square$

Theorem 155. *[Problem 9-9] Suppose M is a smooth manifold and $S \subseteq M$ is an embedded hypersurface (not necessarily compact). Suppose further that there is a smooth vector field V defined on a neighborhood of S and nowhere tangent to S. Then S has a neighborhood in M diffeomorphic to $(-1, 1) \times S$, under a diffeomorphism that restricts to the obvious identification $\{0\} \times S \approx S$.*

Proof. Using Theorem 9.20, we have that $\Phi|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of M. So it remains to show that $\mathcal{O}_\delta = \{(t, p) \in \mathcal{O} : |t| < \delta(p)\}$ is diffeomorphic to $(-1, 1) \times S$. Let $\varphi : (-1, 1) \times S \to \mathcal{O}_\delta$ be given by $(t, p) \mapsto (t\delta(p), p)$; then φ is smooth, and its inverse $\varphi^{-1}(t, p) = (t/\delta(p), p)$ is also smooth. $\square$

Example 156. [Problem 9-10] For each vector field in Example 149, find smooth coordinates in a neighborhood of $(1, 0)$ for which the given vector field is a coordinate vector field.

We only consider the first vector field. We have $V_{(1,0)} = \partial/\partial y$, so we can take S to be the x-axis, parametrized by $X(s) = (s, 0)$. Therefore

$$\Psi(t, s) = \theta_t(s, 0) = \left(s + \frac{1}{2}t^2, t\right),$$

and

$$\Psi^{-1}(x, y) = \left(x - \frac{1}{2}y^2, y\right);$$

$$D\Psi^{-1}(x, y) = \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix}.$$

In these coordinates, we have

$$\begin{aligned} V &= y\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ &= s\frac{\partial}{\partial s} + \left(-s\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right) \\ &= \frac{\partial}{\partial t}. \end{aligned}$$

Chapter 10. Vector Bundles

Theorem 157. *[Exercise 10.1] Suppose E is a smooth vector bundle over M. The projection map $\pi : E \to M$ is a surjective smooth submersion.*

Proof. Let $v \in E$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a smooth local trivialization of E such that $\pi(v) \in U$. Then $\pi_U \circ \Phi = \pi$, so $d\pi_v = d(\pi_U)_{\pi(v)} \circ d\Phi_v$ is surjective since $d\Phi_v$ and $d(\pi_U)_{\pi(v)}$ are both surjective. □

Theorem 158. *[Exercise 10.9] The zero section ζ of every vector bundle $\pi : E \to M$ is continuous, and the zero section of every smooth vector bundle is smooth.*

Proof. Let $p \in M$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialization of E such that $p \in U$. We have $(\Phi \circ \zeta)(x) = (x, 0)$ for all $x \in U$, so $\Phi \circ \zeta|_U$ is continuous and $\zeta|_U$ is continuous since Φ is a homeomorphism. But p was arbitrary, so ζ is continuous. A similar argument holds if π is a smooth vector bundle. □

Theorem 159. *[Exercise 10.11] Let $\pi : E \to M$ be a smooth vector bundle.*

(1) *If $\sigma, \tau \in \Gamma(E)$ and $f, g \in C^\infty(M)$, then $f\sigma + g\tau \in \Gamma(E)$.*
(2) *$\Gamma(E)$ is a module over the ring $C^\infty(M)$.*

Proof. Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialization of E; then

$$(\Phi \circ (f\sigma + g\tau))(x) = (x, f(x)\pi_{\mathbb{R}^k}(\sigma(x)) + g(x)\pi_{\mathbb{R}^k}(\tau(x))),$$

so $f\sigma + g\tau$ is smooth on U. Therefore $f\sigma + g\tau$ is smooth. □

Theorem 160. *[Exercise 10.13] Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Suppose A is a closed subset of M, and $\sigma : A \to E$ is a section of $E|_A$ that is smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each point. For each open subset $U \subseteq M$ containing A, there exists a global smooth section $\widetilde{\sigma} \in \Gamma(E)$ such that $\widetilde{\sigma}|_A = \sigma$ and $\operatorname{supp} \widetilde{\sigma} \subseteq U$.*

Proof. For each $p \in A$, choose a neighborhood W_p of p and a local section $\widetilde{\sigma}_p : W_p \to E$ of E that agrees with σ on $W_p \cap A$. Replacing W_p by $W_p \cap U$ we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{\sigma}_p$ is smooth on W_p by Theorem 159, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \setminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\widetilde{\sigma} : M \to E$ by

$$\widetilde{\sigma}(x) = \sum_{p \in A} \psi_p(x) \widetilde{\sigma}_p(x).$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{\sigma}_p(x) = \sigma(x)$ for each p such that $\psi_p(x) \neq 0$, so

$$\widetilde{\sigma}(x) = \sum_{p \in A} \psi_p(x) \sigma(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) \sigma(x) = \sigma(x),$$

so $\widetilde{\sigma}$ is indeed an extension of σ. It follows from Lemma 1.13(b) that

$$\operatorname{supp} \widetilde{\sigma} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup_{p \in A} \operatorname{supp} \psi_p \subseteq U.$$

$\square$

Theorem 161. *[Exercise 10.14] Let $\pi : E \to M$ be a smooth vector bundle. Every element of E is in the image of a smooth global section.*

Proof. Let $v \in E$ and let $p = \pi(v)$. The assignment $p \mapsto v$ is a section of $E|_{\{p\}}$, which is easily seen to be smooth in the sense of Theorem 160. $\square$

Theorem 162. *[Exercise 10.16] Suppose $\pi : E \to M$ is a smooth vector bundle of rank k.*

(1) *If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, with $1 \le m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.*
(2) *If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$, with $1 \le m \le k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \dots, m$.*
(3) *If $A \subseteq M$ is a closed subset and $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth in the sense described in Theorem 160, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for $i = 1, \dots, k$.*

Proof. Choose a smooth local trivialization $\Phi : \varphi^{-1}(V) \to V \times \mathbb{R}^k$ such that $p \in V$. By shrinking V, we may assume there is a coordinate map $\varphi : V \to \mathbb{R}^n$ and that $V \subseteq U$; let (x^i) be the coordinates. Let $v_i = (\pi_{\mathbb{R}^k} \circ \Phi \circ \sigma_i)(p)$ for $i = 1, \dots, m$, and complete $\{v_1, \dots, v_m\}$ to a basis $\{v_1, \dots, v_k\}$ of $\mathbb{R}^k$. For each $i = m+1, \dots, k$, let $\sigma_i : V \to E$ be the local section of E defined by $\sigma_i(x) = \Phi^{-1}(x, v_i)$. Let W be the subspace of $\mathbb{R}^k$ spanned by $\{v_{m+1}, \dots, v_k\}$; then $F = \Phi^{-1}(V \times (\mathbb{R}^n \setminus W))$ is open, so $V_i = \sigma_i^{-1}(F)$ is a neighborhood of p for each $i = 1, \dots, m$. Let $\widetilde{V} = V \cap V_1 \cap \cdots \cap V_m$. Then $\{\sigma_1, \dots, \sigma_k\}$ is a smooth local frame for E on $\widetilde{V}$ since $(\pi_{\mathbb{R}^k} \circ \Phi \circ \sigma_i)(x) \notin W$ for all $x \in \widetilde{V}$ and $i = 1, \dots, k$. This proves (1). Part (2) follows by choosing constant coefficient local sections σ_i such that $\sigma_i(p) = v_i$ on a sufficiently small neighborhood of p. For (3), apply Theorem 160 to each τ_i and restrict to a sufficiently small neighborhood of A. □

Theorem 163. *[Exercise 10.27] A smooth rank-k vector bundle over M is smoothly trivial if and only if it is smoothly isomorphic over M to the product bundle $M \times \mathbb{R}^k$.*

Proof. If $\pi : E \to M$ is smoothly trivial then there is a trivialization $\Phi : E \to M \times \mathbb{R}^k$. This map satisfies $\pi_M \circ \Phi = \pi$, so E and $M \times \mathbb{R}^k$ are smoothly isomorphic over M. The converse is similar. □

Theorem 164. *[Exercise 10.31] Given a smooth vector bundle $\pi : E \to M$ and a smooth subbundle $D \subseteq E$, the inclusion map $\iota : D \hookrightarrow E$ is a smooth bundle homomorphism over M.*

Proof. This is simply the condition $\pi \circ \iota = \pi|_D$. □

Theorem 165. *[Problem 10-1] The Möbius bundle E is not trivial.*

Proof. E is not homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Removing the center circle $\mathbb{S}^1 \times \{0\}$ from $\mathbb{S}^1 \times \mathbb{R}$ leaves a disconnected space, but removing the center circle from E leaves a connected space. □

Theorem 166. *[Problem 10-2] Let E be a vector bundle over a topological space M. The projection map $\pi : E \to M$ is a homotopy equivalence.*

Proof. Let $\zeta : M \to E$ be the zero section. Then $\pi \circ \zeta = \mathrm{Id}_M$, and $H : E \times I \to E$ defined by $H(v,t) = t\,\mathrm{Id}_E$ is a homotopy from $\zeta \circ \pi$ to Id_E. This shows that π is a homotopy equivalence. □

Theorem 167. *[Problem 10-3] Let* VB *denote the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphisms, and let* Diff *denote the category whose objects are smooth manifolds and whose morphisms are smooth maps. The assignment $M \mapsto TM$, $F \mapsto dF$ defines a covariant functor from* Diff *to* VB*, called the tangent functor.*

Proof. This follows immediately from Corollary 3.22. □

Theorem 168. *[Problem 10-4] The map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ in Lemma 10.5 is smooth.*

Proof. We know that $\sigma : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ given by $\sigma(p, v) = \tau(p)v$ is smooth. Let $\{e_1, \ldots, e_k\}$ be the standard basis for $\mathbb{R}^k$. The (i, j) entry of $\tau(p)$ is $\pi_i(\tau(p)e_j) = \pi_i(\sigma(p, e_j))$, where $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is the projection onto the ith coordinate. This map is smooth as a function of p, so τ is smooth. □

Theorem 169. *[Problem 10-5] Let $\pi : E \to M$ be a smooth vector bundle of rank k over a smooth manifold M with or without boundary. Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M, and for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. For each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, let $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ be the transition function defined by (10.3). The following identity is satisfied for all $\alpha, \beta, \gamma \in A$:*

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Proof. This follows from the fact that

$$\begin{aligned}
(p, \tau_{\alpha\gamma}(p)v) &= (\Phi_\alpha \circ \Phi_\gamma^{-1})(p, v) \\
&= (\Phi_\alpha \circ \Phi_\beta^{-1} \circ \Phi_\beta \circ \Phi_\gamma^{-1})(p, v) \\
&= (p, \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v).
\end{aligned}$$

□

Theorem 170. *[Problem 10-6] Let M be a smooth manifold with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that*

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma$$

is satisfied for all $\alpha, \beta, \gamma \in A$. Then there is a smooth rank-k vector bundle $E \to M$ with smooth local trivializations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose transition functions are the given maps $\tau_{\alpha\beta}$.

Proof. Note that $\tau_{\alpha\alpha} = \mathrm{Id}_{\mathbb{R}^k}$ for all $\alpha \in A$. Let $F = \coprod_{\alpha \in A} U_\alpha \times \mathbb{R}^k$ and let $i_\alpha : U_\alpha \times \mathbb{R}^k \to F$ be the canonical injection for each $\alpha \in A$. Define an equivalence relation $\sim$ on F by declaring that $(p, v)_\alpha \sim (p', v')_\beta$ if and only if $p = p'$ and $\tau_{\alpha\beta}(p)v' = v$. This relation is symmetric because

$$\tau_{\alpha\beta}(p)\tau_{\beta\alpha}(p) = \tau_{\alpha\alpha}(p) = \mathrm{Id}_{\mathbb{R}^k},$$

so $\tau_{\alpha\beta}(p)v' = v$ implies that $v' = \tau_{\beta\alpha}(p)v$. Similarly, the relation is transitive because $\tau_{\alpha\beta}(p)v' = v$ and $\tau_{\beta\gamma}(p)v'' = v'$ implies that

$$\tau_{\alpha\gamma}(p)v'' = \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v'' = \tau_{\alpha\beta}(p)v' = v.$$

Let $E = F/\sim$ and let $q : F \to E$ be the quotient map. Let $E_p = q\left(\{p\} \times \mathbb{R}^k\right)$ for each $p \in M$; we want to give E_p a vector space structure. Define $\widetilde{m} : \mathbb{R} \times q^{-1}(E_p) \to E_p$ by $m(r, (p, v)_\alpha) = q((p, rv)_\alpha)$. If $(p, v)_\alpha \sim (p, v')_\beta$ then $\tau_{\alpha\beta}(p)v' = v$ and $\tau_{\alpha\beta}(p)rv' = r\tau_{\alpha\beta}(p)v' = rv$. Therefore $\widetilde{m}$ descends to a continuous map $m : \mathbb{R} \times E_p \to E_p$, which is scalar multiplication in E_p. A similar construction shows that vector addition can be defined on E_p.

Let $\widetilde{\pi} : F \to M$ be given by $(p, v) \mapsto p$; then $\widetilde{\pi}$ descends to a continuous map $\pi : E \to M$ satisfying $\pi \circ q = \widetilde{\pi}$. For each $\alpha \in A$, define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ as the unique bijection satisfying $\Phi_\alpha \circ q \circ i_\alpha = \mathrm{Id}_{U_\alpha \times \mathbb{R}^k}$. We now check conditions (ii) and (iii) in Lemma 10.6. For (ii), it is clear that $\Phi_\alpha|_{E_p}$ is linear, so it is an isomorphism. For (iii), we have

$$\begin{aligned}(\Phi_\alpha \circ \Phi_\beta^{-1})(p, v) &= \Phi_\alpha(q((p, v)_\beta)) \\ &= \Phi_\alpha(q((p, \tau_{\alpha\beta}(p)v)_\alpha)) \\ &= (p, \tau_{\alpha\beta}(p)v).\end{aligned}$$

Lemma 10.6 now shows that $\pi : E \to M$ is a smooth rank-k vector bundle, with $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations. □

Example 171. [Problem 10-7] Compute the transition function for $T\mathbb{S}^2$ associated with the two local trivializations determined by stereographic coordinates.

See Theorem 128.

Theorem 172. *[Problem 10-8] Let Vec_1 be the category whose objects are finite-dimensional real vector spaces and whose morphisms are linear isomorphisms. If $\mathcal{F}$ is a covariant functor from Vec_1 to itself, for each finite-dimensional vector space V we get a map $\mathcal{F} : \mathrm{GL}(V) \to \mathrm{GL}(\mathcal{F}(V))$ sending each isomorphism $A : V \to V$ to the induced isomorphism $\mathcal{F}(A) : \mathcal{F}(V) \to \mathcal{F}(V)$. We say $\mathcal{F}$ is a **smooth functor** if this map is smooth for every V. Given a smooth vector bundle $\pi : E \to M$ and a smooth functor*

$\mathcal{F} : \mathsf{Vec}_1 \to \mathsf{Vec}_1$, *there is a smooth vector bundle* $\widetilde{\pi} : \mathcal{F}(E) \to M$ *whose fiber at each point* $p \in M$ *is* $\mathcal{F}(E_p)$.

Proof. Let k be the rank of π. Define $\mathcal{F}(E) = \coprod_{p\in M} \mathcal{F}(E_p)$ and let $\widetilde{\pi} : \mathcal{F}(E) \to M$ be the projection. Let $\{U_p\}_{p\in M}$ be an open cover of M where each U_p contains p and has an associated local trivialization $\Phi_p : \pi^{-1}(U_p) \to U_p \times \mathbb{R}^k$. Choose an arbitrary isomorphism $\varphi : \mathcal{F}(\mathbb{R}^k) \cong \mathbb{R}^n$, where n is the dimension of $\mathcal{F}(\mathbb{R}^k)$. For each $p \in M$, define $\widetilde{\Phi}_p : \widetilde{\pi}^{-1}(U_p) \to U_p \times \mathbb{R}^n$ as follows: for each $x \in U_p$ and $v \in \widetilde{E}_x = \widetilde{\pi}^{-1}(\{x\})$, set $\widetilde{\Phi}_p(v) = (x, (\varphi \circ \mathcal{F}(\Phi_p|_{E_x}))(v))$ where

$$\mathcal{F}(\Phi_p|_{E_x}) : \mathcal{F}(E_x) \to \mathcal{F}(\{x\} \times \mathbb{R}^k) \cong \mathcal{F}(\mathbb{R}^k)$$

is an isomorphism. For each $p, q \in M$ with $U_p \cap U_q \neq \emptyset$, we have

$$\begin{aligned}(\widetilde{\Phi}_p \circ \widetilde{\Phi}_q^{-1})(x, v) &= (x, (\varphi \circ \mathcal{F}(\Phi_p|_{E_x}) \circ \mathcal{F}(\Phi_q|_{E_x})^{-1} \circ \varphi^{-1})(v)) \\ &= (x, (\varphi \circ \mathcal{F}(\Phi_p|_{E_x} \circ (\Phi_q|_{E_x})^{-1}) \circ \varphi^{-1})(v)) \\ &= (x, \tau_{pq}(x)v)\end{aligned}$$

for some smooth τ_{pq} since $\mathcal{F}$ is a smooth functor. Applying Lemma 10.6 completes the proof. □

Theorem 173. *[Problem 10-9] Suppose* M *is a smooth manifold,* $E \to M$ *is a smooth vector bundle, and* $S \subseteq M$ *is an embedded submanifold with or without boundary. For any smooth section* σ *of the restricted bundle* $E|_S \to S$*, there exist a neighborhood* U *of* S *in* M *and a smooth section* $\widetilde{\sigma}$ *of* $E|_U$ *such that* $\sigma = \widetilde{\sigma}|_S$*. If* E *has positive rank, then every smooth section of* $E|_S$ *extends smoothly to all of* M *if and only if* S *is properly embedded.*

Proof. As in Theorem 82. □

Theorem 174. *[Problem 10-11] Suppose* E *and* E' *are smooth vector bundles over a smooth manifold* M *with or without boundary, and* $F : E \to E'$ *is a bijective smooth bundle homomorphism over* M*. Then* F *is a smooth bundle isomorphism.*

Proof. Let k be the rank of E and let k' be the rank of E'; let $\pi : E \to M$ and $\pi' : E' \to M$ be the projections. We want to show that F^{-1} is smooth. Let $v \in E$ and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a smooth local trivialization of E satisfying $\pi(v) \in U$ and $\pi_U \circ \Phi = \pi$. Let $\Phi' : (\pi')^{-1}(U') \to U' \times \mathbb{R}^{k'}$ be a smooth local trivialization of E' satisfying $\pi'(F(v)) \in U'$ and $\pi_{U'} \circ \Phi' = \pi'$. Consider the smooth map $f = \Phi' \circ F \circ \Phi^{-1}$ with its inverse $f^{-1} = \Phi \circ F^{-1} \circ (\Phi')^{-1}$ defined on a sufficiently small neighborhood $\widetilde{U}' \times \mathbb{R}^{k'}$ of $\Phi'(F(v))$. For each $x \in U$ the map $f|_{\{x\}\times\mathbb{R}^k}$ is an isomorphism whose inverse is $f^{-1}|_{\{y(x)\}\times\mathbb{R}^{k'}}$, where $y(x) = \pi'(F(\zeta(x)))$ and $\zeta : M \to E$ is the zero section. Since operator inversion is smooth and y is smooth, f^{-1} is smooth on $\widetilde{U}' \times \mathbb{R}^{k'}$. Therefore F^{-1} is smooth on a neighborhood of $F(v)$, and since v was arbitrary, F^{-1} is smooth. $\square$

Theorem 175. *[Problem 10-12] Let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M with or without boundary. Suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit smooth local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}$ and $\{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$, respectively. Then E and $\widetilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k,\mathbb{R})$ such that*

$$(*) \qquad \widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \qquad p \in U_\alpha \cap U_\beta.$$

Proof. Suppose there is a smooth bundle isomorphism $F : E \to \widetilde{E}$. Let $\alpha \in A$ and let $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and $\widetilde{\Phi}_\alpha : \widetilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ be the associated smooth local trivializations. Since $\pi_{U_\alpha} \circ \widetilde{\Phi}_\alpha \circ F = \widetilde{\pi} \circ F = \pi$, the map $\widetilde{\Phi}_\alpha \circ F : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a smooth local trivialization of E. Therefore

$$(\widetilde{\Phi}_\alpha \circ F \circ \Phi_\alpha^{-1})(p,v) = (p, \sigma_\alpha(p)v)$$

for some smooth linear map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k,\mathbb{R})$. Now let $\alpha, \beta \in A$. We have

$$\begin{aligned}
(p, \widetilde{\tau}_{\alpha\beta}(p)v) &= (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})(p,v) \\
&= (\widetilde{\Phi}_\alpha \circ F \circ \Phi_\alpha^{-1} \circ \Phi_\alpha \circ \Phi_\beta^{-1} \circ \Phi_\beta \circ F^{-1} \circ \widetilde{\Phi}_\beta^{-1})(p,v) \\
&= (p, \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}v),
\end{aligned}$$

so $\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}$. Conversely, suppose that (*) holds. For each $\alpha \in A$, define a map $F_\alpha : \pi^{-1}(U_\alpha) \to \widetilde{\pi}^{-1}(U_\alpha)$ by $F_\alpha = \widetilde{\Phi}_\alpha^{-1} \circ \varphi_\alpha \circ \Phi_\alpha$, where $\varphi_\alpha(p,v) = (p, \sigma_\alpha(p)v)$. If $\alpha, \beta \in A$ and $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\begin{aligned}
F_\beta &= \widetilde{\Phi}_\beta^{-1} \circ \varphi_\beta \circ \Phi_\beta \\
&= \widetilde{\Phi}_\alpha^{-1} \circ \widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1} \circ \varphi_\beta \circ \Phi_\beta \circ \Phi_\alpha^{-1} \circ \Phi_\alpha \\
&= \widetilde{\Phi}_\alpha^{-1} \circ \varphi_\alpha \circ \Phi_\alpha \\
&= F_\alpha
\end{aligned}$$

on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ by (*), so the gluing lemma shows that there is a smooth map $F : E \to \widetilde{E}$ such that $F|_{\pi^{-1}(U_\alpha)} = F_\alpha$ for all $\alpha \in A$. The restriction of F to each fiber is clearly linear. On any $\pi^{-1}(U_\alpha)$ we have

$$\begin{aligned}
\widetilde{\pi} \circ F &= \widetilde{\pi} \circ \widetilde{\Phi}_\alpha^{-1} \circ \varphi_\alpha \circ \Phi_\alpha \\
&= \pi_{U_\alpha} \circ \varphi_\alpha \circ \Phi_\alpha \\
&= \pi_{U_\alpha} \circ \Phi_\alpha
\end{aligned}$$

$$= \pi,$$

so $\widetilde{\pi} \circ F = \pi$ on E. By Theorem 174, F is a smooth bundle isomorphism. □

Theorem 176. *[Problem 10-14] Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. Identifying T_pS as a subspace of T_pM for each $p \in S$ in the usual way, TS is a smooth subbundle of $TM|_S$.*

Proof. Let $\iota : S \hookrightarrow M$ be the inclusion map. For each $p \in S$, let $D_p = d\iota_p(T_pS)$. Let $D = \bigcup_{p \in S} D_p = TS \subseteq TM|_S$. Let $p \in S$, let V be a neighborhood of p in S that is an embedded submanifold of M, and let (U, φ) be a slice chart for V containing p. Write $\varphi = (x^1, \ldots, x^n)$ and let k be the dimension of S. Then $V \cap U$ is neighborhood of p in S, and the maps $\sigma_i : V \cap U \to TM|_S$ given by $\sigma_i(q) = d\varphi_q(\partial/\partial x^i|_q)$ for $i = 1, \ldots, k$ satisfy the condition in Lemma 10.32. □

Theorem 177. *[Problem 10-17] Suppose $M \subseteq \mathbb{R}^n$ is an immersed submanifold. The ambient tangent bundle $T\mathbb{R}^n|_M$ is isomorphic to the Whitney sum $TM \oplus NM$, where $NM \to M$ is the normal bundle.*

Proof. Obvious. □

Theorem 178. *[Problem 10-19] Suppose $\pi : E \to M$ is a fiber bundle with fiber F.*

(1) *π is an open quotient map.*
(2) *If the bundle is smooth, then π is a smooth submersion.*
(3) *π is a proper map if and only if F is compact.*
(4) *E is compact if and only if both M and F are compact.*

Proof. For (1), let V be open in E. Let $x \in \pi(V)$ and let $\Phi : \pi^{-1}(U) \to U \times F$ be a local trivialization of E such that $x \in U$. Choose any $f \in F$ and a neighborhood $\widetilde{V} \subseteq \pi^{-1}(U)$ of $\Phi^{-1}(x, f)$. Then $\Phi(\widetilde{V})$ is open in $U \times F$ since Φ is a homeomorphism, and $\pi_U(\Phi(\widetilde{V})) = \pi(\widetilde{V})$ is a neighborhood of x contained in $\pi(V)$. This shows that π is open, and π is a quotient map since it is surjective. For (2), let $v \in E$ and let $\Phi : \pi^{-1}(U) \to U \times F$ be a smooth local trivialization of E such that $\pi(v) \in U$. Then $\pi_U \circ \Phi = \pi$, so $d\pi_v = d(\pi_U)_{\pi(v)} \circ d\Phi_v$ is surjective since $d\Phi_v$ and $d(\pi_U)_{\pi(v)}$ are both surjective. This shows that π is a smooth submersion. For (3), suppose that π is proper. If $x \in M$ then $\pi^{-1}(\{x\}) \approx \{x\} \times F \approx F$ is compact. Conversely, suppose that F is compact and $A \subseteq M$ is compact. For each $p \in A$, choose a local trivialization $\Phi : \pi^{-1}(U_p) \to U_p \times F$ such that $p \in U_p$, and choose a precompact neighborhood V_p of p such that $\overline{V_p} \subseteq U_p$. Since A is compact, the open cover $\{V_p\}$ has a finite subcover

$\{V_{p_1}, \ldots, V_{p_k}\}$. Then $\pi^{-1}(A)$ is a closed subset of

$$\bigcup_{i=1}^{k} \pi^{-1}(\overline{V_{p_i}}) \cong \bigcup_{i=1}^{k} \overline{V_{p_i}} \times F,$$

which is compact. Therefore $\pi^{-1}(A)$ is compact. For (4), if E is compact then $M = \pi(E)$ is compact and F is compact by (3) since π is proper. Conversely, if M and F are compact then π is proper by (3), so $E = \pi^{-1}(M)$ is compact. □

Chapter 11. The Cotangent Bundle

Theorem 179. *[Exercise 11.10] Suppose M is a smooth manifold and $E \to M$ is a smooth vector bundle over M. Define the **dual bundle** to E to be the bundle $E^* \to M$ whose total space is the disjoint union $E^* = \coprod_{p \in M} E_p^*$, where E_p^* is the dual space to E_p, with the obvious projection. Then $E^* \to M$ is a smooth vector bundle, whose transition functions are given by $\tau^*(p) = (\tau(p)^{-1})^T$ for any transition function $\tau : U \to \mathrm{GL}(k, \mathbb{R})$ of E.*

Proof. Since

$$\begin{aligned} \tau^*_{\alpha\beta}(p)\tau^*_{\beta\gamma}(p) &= (\tau_{\alpha\beta}(p)^{-1})^T (\tau_{\beta\gamma}(p)^{-1})^T \\ &= ((\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p))^{-1})^T \\ &= \tau^*_{\alpha\gamma}(p), \end{aligned}$$

we can apply Theorem 170. □

Theorem 180. *[Exercise 11.12] Let M be a smooth manifold with or without boundary, and let $\omega : M \to T^*M$ be a rough covector field.*

1. *ω is smooth.*
2. *In every smooth coordinate chart, the component functions of ω are smooth.*
3. *Each point of M is contained in some coordinate chart in which ω has smooth component functions.*
4. *For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.*
5. *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth on U.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. If (3) holds for a coordinate chart $(U, (x^i))$ and the natural coordinates (x^i, ξ_i) on $\pi^{-1}(U) \subseteq T^*M$ then the coordinate representation of ω on U is

$$\widehat{\omega}(x) = (x^1, \ldots, x^n, \omega_1(x), \ldots, \omega_n(x)),$$

which is smooth if $\omega_1, \ldots, \omega_n$ are smooth. This proves (3) $\Rightarrow$ (1). This also implies (3) $\Rightarrow$ (4). Suppose that (4) holds, $U \subseteq M$ is open and X is a smooth vector field on U.

For each $p \in U$, choose a smooth bump function ψ that is equal to 1 on a neighborhood of p and supported in U. Define $\widetilde{X} = \psi X$, extended to be zero on $M \setminus \operatorname{supp} \psi$. Then $\omega(\widetilde{X})$ is smooth and is equal to $\omega(X)$ in a neighborhood of p. So $\omega(X)$ is smooth in a neighborhood of every point in U, which proves (4) $\Rightarrow$ (5). If (5) holds then (2) holds, since $\omega_i = \omega(\partial/\partial x^i)$ for any smooth chart $(U, (x^i))$. $\square$

Theorem 181. *[Exercise 11.16] Let M be a smooth manifold with or without boundary, and let ω be a rough covector field on M. If (ε^i) is a smooth coframe on an open subset $U \subseteq M$, then ω is smooth on U if and only if its component functions with respect to (ε^i) are smooth.*

Proof. This follows from Proposition 10.22. $\square$

Example 182. [Exercise 11.17] Let $f(x, y) = x^2$ on $\mathbb{R}^2$, and let X be the vector field

$$X = \operatorname{grad} f = 2x \frac{\partial}{\partial x}.$$

Compute the coordinate expression for X in polar coordinates (on some open subset on which they are defined) and show that it is *not* equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

Set $x = r \cos \theta$ and $y = r \sin \theta$; then $f(r, \theta) = r^2 \cos^2 \theta$, so

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} = 2r \cos^2 \theta \frac{\partial}{\partial r} - 2r^2 \sin \theta \cos \theta \frac{\partial}{\partial \theta}.$$

The derivative of the transition map $\varphi(r, \theta) = (r \cos \theta, r \sin \theta)$ is

$$\begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}.$$

Therefore

$$X = 2r \cos \theta \left(\cos \theta \frac{\partial}{\partial r} + \sin \theta \frac{\partial}{\partial \theta} \right) \neq \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

Theorem 183. *[Exercise 11.21] Let M be a smooth manifold with or without boundary, and let $f, g \in C^\infty(M)$.*

(1) *If a and b are constants, then $d(af + bg) = a\,df + b\,dg$.*
(2) *$d(fg) = f\,dg + g\,df$.*
(3) *$d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.*
(4) *If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\,df$.*
(5) *If f is constant, then $df = 0$.*

Proof. For (1), we have

$$d(af+bg)_p(v) = v(af+bg) = avf + bvg = (a\,df_p + b\,dg_p)(v).$$

For (2), we have

$$d(fg)_p(v) = v(fg) = f(p)vg + g(p)vf = (f\,dg_p + g\,df_p)(v).$$

For (3), if $p \in M$ and $(U,(x^i))$ is a smooth chart containing p then on U,

$$\begin{aligned} d(f/g)_x &= \sum_{i=1}^n \frac{\partial (f/g)}{\partial x^i}(x)\,dx^i|_x \\ &= \frac{1}{g(x)^2}\sum_{i=1}^n \left[g(x)\frac{\partial f}{\partial x^i}(x) - f(x)\frac{\partial g}{\partial x^i}(x) \right] dx^i|_x \\ &= (g(x)\,df_x - f(x)\,dg_x)/g(x)^2. \end{aligned}$$

For (4), we have

$$d(h\circ f)_p(v) = dh_{f(p)}(df_p(v)) = h'(f(p))df_p(v).$$

For (5), if $(U,(x^i))$ is a smooth chart and $\widehat{f}$ is the coordinate representation of f on U then $D\widehat{f} = 0$, so $\widehat{f}$ is constant and f is constant on U. The result follows from Theorem 29. □

Theorem 184. *[Exercise 11.24] For a smooth real-valued function $f : M \to \mathbb{R}$, $p \in M$ is a critical point of f if and only if $df_p = 0$.*

Proof. Obvious. □

Theorem 185. *[Exercise 11.35] Let M be a smooth manifold with or without boundary. Suppose $\gamma : [a,b] \to M$ is a piecewise smooth curve segment, and $\omega, \omega_1, \omega_2 \in \mathfrak{X}^*(M)$.*

(1) *For any $c_1, c_2 \in \mathbb{R}$,*
$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1\int_\gamma \omega_1 + c_2\int_\gamma \omega_2.$$
(2) *If γ is a constant map, then $\int_\gamma \omega = 0$.*
(3) *If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ with $a < c < b$, then*
$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$
(4) *If $F : M \to N$ is any smooth map and $\eta \in \mathfrak{X}^*(N)$, then*
$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Proof. We can assume that γ is smooth rather than piecewise smooth. For (1),

$$\begin{aligned}\int_\gamma (c_1\omega_1 + c_2\omega_2) &= \int_{[a,b]} \gamma^*(c_1\omega_1 + c_2\omega_2) \\ &= c_1 \int_{[a,b]} \gamma^*\omega_1 + c_2 \int_{[a,b]} \gamma^*\omega_2 \\ &= c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2\end{aligned}$$

since γ^* is linear. For (2), if γ is constant then $d\gamma = 0$ and $\gamma^* = 0$. For (3),

$$\begin{aligned}\int_\gamma \omega &= \int_{[a,b]} \gamma^*\omega \\ &= \int_{[a,c]} \gamma^*\omega + \int_{[c,b]} \gamma^*\omega \\ &= \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.\end{aligned}$$

For (4),

$$\int_\gamma F^*\eta = \int_{[a,b]} \gamma^* F^*\eta = \int_{[a,b]} (F \circ \gamma)^*\eta = \int_{F\circ\gamma} \eta.$$

$\square$

Theorem 186. *[Exercise 11.41] A smooth covector field ω is conservative if and only if its line integrals are path-independent, in the sense that $\int_\gamma \omega = \int_{\widetilde{\gamma}} \omega$ whenever γ and $\widetilde{\gamma}$ are piecewise smooth curve segments with the same starting and ending points.*

Proof. Suppose that ω is conservative, γ is defined on $[a, b]$, and $\widetilde{\gamma}$ is defined on $[c, d]$. Define $\gamma - \widetilde{\gamma} : [a, b + d - c] \to M$ by

$$(\gamma - \widetilde{\gamma})(t) = \begin{cases} \gamma(t), & 0 \le t \le b, \\ \widetilde{\gamma}(b + d - t), & b \le t \le b + d - c. \end{cases}$$

Then $\gamma - \widetilde{\gamma}$ is piecewise smooth, so

$$0 = \int_{\gamma - \widetilde{\gamma}} \omega = \int_\gamma \omega - \int_{\widetilde{\gamma}} \omega$$

by Theorem 185. The converse is obvious since any piecewise smooth closed curve segment has the same starting and ending points as a constant curve segment. $\square$

Theorem 187. *[Problem 11-1]*

(1) *Suppose V and W are finite-dimensional vector spaces and $A : V \to W$ is any linear map. The following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\xi_V"'] & W \arrow[d, "\xi_W"] \\ V^{**} \arrow[r, "(A^*)^*"'] & W^{**} \end{tikzcd}$$

where ξ_V and ξ_W denote the natural isomorphisms.

(2) *There does not exist a way to assign to each finite-dimensional vector space V an isomorphism $\beta_V : V \to V^*$ such that for every linear map $A : V \to W$, the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\beta_V"'] & W \arrow[d, "\beta_W"] \\ V^{*} & W^{*} \arrow[l, "A^*"] \end{tikzcd}$$

Proof. For (1), we have

$$((A^*)^*(\xi_V v))\omega = (\xi_V v)(A^*\omega) = (A^*\omega)v = \omega(Av) = (\xi_W(Av))\omega.$$

For (2), suppose that there are such isomorphisms β_V. Take $V = W = \mathbb{R}$. The diagram commutes when $A = 0$, so

$$\beta_{\mathbb{R}} = 0 \circ \beta_{\mathbb{R}} \circ 0 = 0.$$

This contradicts the fact that $\beta_{\mathbb{R}}$ is an isomorphism. □

Theorem 188. *[Problem 11-3] Let Vec_1 be the category of finite-dimensional vector spaces and linear isomorphisms as in Theorem 172. Define a functor $\mathcal{F} : \mathsf{Vec}_1 \to \mathsf{Vec}_1$ by setting $\mathcal{F}(V) = V^*$ for a vector space V, and $\mathcal{F}(A) = (A^{-1})^*$ for an isomorphism A. Then $\mathcal{F}$ is a smooth covariant functor, and for every M, $\mathcal{F}(TM)$ and T^*M are canonically smoothly isomorphic vector bundles.*

Proof. $\mathcal{F}$ is covariant because

$$((A \circ B)^{-1})^* = (B^{-1} \circ A^{-1})^* = (A^{-1})^* \circ (B^{-1})^*,$$

and it is smooth because matrix inversion and transposition are smooth. So Theorem 172 shows that $\mathcal{F}(TM)$ exists, and we can define a smooth vector bundle isomorphism $\varphi : \mathcal{F}(TM) \to T^*M$ by setting $\varphi|_{T_p^*M} = \mathrm{Id}_{T_p^*M}$ for each $p \in M$. □

Theorem 189. *[Problem 11-4] Let M be a smooth manifold with or without boundary and let p be a point of M. Let $\mathcal{J}_p$ denote the subspace of $C^\infty(M)$ consisting of smooth* *functions that vanish at p, and let $\mathcal{J}_p^2$ be the subspace of $\mathcal{J}_p$ spanned by functions of the form fg for some $f, g \in \mathcal{J}_p$.*

(1) *$f \in \mathcal{J}_p^2$ if and only if in any smooth local coordinates, its first-order Taylor polynomial at p is zero.*
(2) *Define a map $\Phi : \mathcal{J}_p \to T_p^*M$ by setting $\Phi(f) = df_p$. The restriction of Φ to $\mathcal{J}_p^2$ is zero, and Φ descends to a vector space isomorphism from $\mathcal{J}_p/\mathcal{J}_p^2$ to T_p^*M.*

Proof. If $f \in \mathcal{J}_p^2$ then $f = \sum_{i=1}^k \alpha_i g_i h_i$ for some $\alpha_i \in \mathbb{R}$ and $g_i, h_i \in \mathcal{J}_p$. If in smooth local coordinates we have $Df_1(p) = Df_2(p) = 0$, then

$$D(g_i h_i)(p)u = g_i(p)[Dh_i(p)u] + [Dg_i(p)u]h_i(p) = 0,$$

so the first-order Taylor polynomial of f is zero by linearity. Conversely, if $Df(p) = 0$ then Theorem C.15 shows that locally we can write

$$f(x) = \sum_{i,j=1}^n (x^i - p^i)(x^j - p^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(p + t(x-p))\, dt,$$

which is an element of $\mathcal{J}_p^2$ since $x^i - p^i$ and $x^j - p^j$ times the integral are elements of $\mathcal{J}_p$. Therefore $f = \sum_{i=1}^k \alpha_i g_i h_i$ for some $\alpha_i \in \mathbb{R}$ and $g_i, h_i \in \mathcal{J}_p$ in a neighborhood U of p. Choose a precompact neighborhood V of p with $\overline{V} \subseteq U$ and choose a partition of unity $\{\psi_0, \psi_1\}$ subordinate to the open cover $\{U, M \setminus \overline{V}\}$. We take $g_i = h_i = 0$ on $M \setminus \operatorname{supp} \psi_0$. Then $\psi_1, f \in \mathcal{J}_p$ and

$$f = \sum_{i=1}^k \alpha_i (\psi_0 g_i) h_i + \psi_1 f,$$

which proves (1). Part (2) follows immediately. □

Theorem 190. *[Problem 11-5] For any smooth manifold M, T^*M is a trivial vector bundle if and only if TM is trivial.*

Proof. If TM is trivial then it admits a smooth global frame; Lemma 11.14 shows that its dual coframe is a smooth global coframe for M, so T^*M is trivial. The converse is similar. □

Theorem 191. *[Problem 11-6] Suppose M is a smooth n-manifold, $p \in M$, and $y^1, \ldots, y^k$ are smooth real-valued functions defined on a neighborhood of p in M.*

(1) *If $k = n$ and $(dy^1|_p, \ldots, dy^n|_p)$ is a basis for T_p^*M then $(y^1, \ldots, y^n)$ are smooth coordinates for M in some neighborhood of p.*
(2) *If $(dy^1|_p, \ldots, dy^k|_p)$ is a linearly independent k-tuple of covectors and $k < n$, then there are smooth functions $y^{k+1}, \ldots, y^n$ such that $(y^1, \ldots, y^n)$ are smooth coordinates for M in a neighborhood of p.*

(3) *If* $(dy^1|_p, \dots, dy^k|_p)$ *span* $T_p^* M$*, there are indices* $i_1, \dots, i_n$ *such that* $(y^{i_1}, \dots, y^{i_n})$ *are smooth coordinates for* M *in a neighborhood of* p*.*

Proof. Part (1) follows from the inverse function theorem. For (2), complete $(dy^1|_p, \dots, dy^k|_p)$ to a basis $(dy^1|_p, \dots, dy^k|_p, \omega^{k+1}, \dots, \omega^n)$, choose smooth functions y^i such that $dy^i|_p = \omega^i$ for $i = k+1, \dots, n$, and apply (1). For (3), choose indices $i_1, \dots, i_n$ such that $dy^{i_1}|_p, \dots, dy^{i_n}|_p$ is a basis for $T_p^* M$ and apply (1). □

Example 192. [Problem 11-7] In the following problems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and $\omega \in \mathfrak{X}^*(N)$. Compute $F^*\omega$ in each case.

(1) $M = N = \mathbb{R}^2$; $F(s,t) = (st, e^t)$; $\omega = x\,dy - y\,dx$.
(2) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$; $F(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$; $\omega = z^2\,dx$.
(3) $M = \{(s,t) \in \mathbb{R}^2 : s^2 + t^2 < 1\}$ and $N = \mathbb{R}^3 \setminus \{0\}$; $F(s,t) = (s, t, \sqrt{1 - s^2 - t^2})$; $\omega = (1 - x^2 - y^2)\,dz$.

For (1),

$$dx = t\,ds + s\,dt, \quad dy = e^t\,dt,$$

so

$$\omega = st(e^t\,dt) - e^t(t\,ds + s\,dt) = -te^t\,ds + se^t(t-1)\,dt.$$

For (2),

$$dx = -(\cos\varphi + 2)\sin\theta\,d\theta - \sin\varphi\cos\theta\,d\varphi,$$

so

$$\omega = \sin^2\varphi(-(\cos\varphi + 2)\sin\theta\,d\theta - \sin\varphi\cos\theta\,d\varphi).$$

For (3),

$$\begin{aligned} dx &= ds, \\ dy &= dt, \\ dz &= \frac{-s}{\sqrt{1 - s^2 - t^2}}\,ds + \frac{-t}{\sqrt{1 - s^2 - t^2}}\,dt, \end{aligned}$$

so

$$\omega = -s\sqrt{1 - s^2 - t^2}\,ds - t\sqrt{1 - s^2 - t^2}\,dt.$$

Theorem 193. *[Problem 11-8]*

(1) *Suppose* $F : M \to N$ *is a diffeomorphism, and let* $dF^* : T^*N \to T^*M$ *be the map whose restriction to each cotangent space* $T_q^* N$ *is equal to* $dF^*_{F^{-1}(q)}$*. Then* dF^* *is a smooth bundle homomorphism.*

(2) *Let* Diff_1 *be the category whose objects are smooth manifolds, but whose only morphisms are diffeomorphisms; and let* VB *be the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphisms. The assignment* $M \mapsto T^*M$*,* $F \mapsto dF^*$ *defines a contravariant functor from* Diff_1 *to* VB*, called the* ***cotangent functor***.

Proof. It is clear that dF^* is a smooth bundle homomorphism covering F^{-1}. Since

$$d(F \circ G)_p^* = (dF_{G(p)} \circ dG_p)^* = dG_p^* \circ dF_{G(p)}^*$$

we have $d(F \circ G)^* = dG^* \circ dF^*$. Also, $d(\mathrm{Id}_M)^* = \mathrm{Id}_{T^*M}$. This shows that $M \mapsto T^*M$, $F \mapsto dF^*$ defines a contravariant functor. $\square$

Example 194. [Problem 11-9] Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = x^2 + y^2 + z^2$, and let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the following map (the inverse of the stereographic projection):

$$F(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Compute F^*df and $d(f \circ F)$ separately, and verify that they are equal.

We have

$$df = 2x\,dx + 2y\,dy + 2z\,dz$$

and

$$\begin{aligned} dx &= \frac{2(u^2+v^2+1) - 4u^2}{(u^2+v^2+1)^2}\,du + \frac{-4uv}{(u^2+v^2+1)^2}\,dv, \\ dy &= \frac{-4uv}{(u^2+v^2+1)^2}\,du + \frac{2(u^2+v^2+1) - 4v^2}{(u^2+v^2+1)^2}\,dv, \\ dz &= \frac{4u}{(u^2+v^2+1)^2}\,du + \frac{4v}{(u^2+v^2+1)^2}\,dv, \end{aligned}$$

so

$$df = 0.$$

Alternatively we have $f \circ F = 1$, so $d(f \circ F) = 0$.

Example 195. [Problem 11-10] In each of the cases below, M is a smooth manifold and $f : M \to \mathbb{R}$ is a smooth function. Compute the coordinate representation for df, and determine the set of all points $p \in M$ at which $df_p = 0$.

(1) $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$; $f(x, y) = x/(x^2 + y^2)$. Use standard coordinates (x, y).
(2) M and f are as in (1); this time use polar coordinates (r, θ).
(3) $M = \mathbb{S}^2 \subseteq \mathbb{R}^3$; $f(p) = z(p)$ (the z-coordinate of p as a point in $\mathbb{R}^3$). Use north and south stereographic coordinates.
(4) $M = \mathbb{R}^n$; $f(x) = |x|^2$. Use standard coordinates.

For (1),
$$df = \frac{-x^2+y^2}{(x^2+y^2)^2}\,dx + \frac{-2xy}{(x^2+y^2)^2}\,dy,$$
which is zero when $x^2=y^2$ and $xy=0$, i.e. never on M. For (2) we have
$$f(r,\theta) = \frac{\cos\theta}{r},$$
so
$$df = -\frac{\cos\theta}{r^2}\,dr - \frac{\sin\theta}{r}\,d\theta,$$
which is zero when $\cos\theta = \sin\theta = 0$, i.e. never on M. For (3) we have
$$f(u,v) = \frac{u^2+v^2-1}{u^2+v^2+1}$$
on $\mathbb{S}^2 \setminus N$ using the north stereographic projection, so
$$df = \frac{4u}{(u^2+v^2+1)^2}\,du + \frac{4v}{(u^2+v^2+1)^2}\,dv,$$
which is zero when $u=v=0$, i.e. when p is the south pole $(0,0,-1)$. A similar calculation using the south stereographic projection shows that $df_p = 0$ when p is the north pole $(0,0,1)$. For (4) we have
$$df = 2x^1\,dx^1 + \cdots + 2x^n\,dx^n,$$
which is zero when $p=0$.

Theorem 196. *[Problem 11-11] Let M be a smooth manifold, and $C \subseteq M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Given a smooth local defining function $\Phi : U \to \mathbb{R}^k$ for C on a neighborhood U of p in M, there are real numbers $\lambda_1, \ldots, \lambda_k$ (called **Lagrange multipliers**) such that*
$$df_p = \lambda_1 d\Phi^1|_p + \cdots + \lambda_k d\Phi^k|_p.$$

Proof. Let n be the dimension of M; then C has dimension $n-k$. Let (V,φ) be a smooth slice chart for C in M centered at p such that $V \subseteq U$, let $\widetilde{f} = f \circ \varphi^{-1}$, let $\widetilde{\Phi} = \Phi \circ \varphi^{-1}$, and let $\widetilde{p} = \varphi(p)$. It suffices to show that there are real numbers $\lambda_1, \ldots, \lambda_k$ such that
$$D\widetilde{f}(p) = \lambda_1 D\widetilde{\Phi}^1(p) + \cdots + \lambda_k D\widetilde{\Phi}^k(p).$$
Since $\Phi^{-1}(0) \subseteq V \cap C$, this follows from the method of Lagrange multipliers on $\mathbb{R}^n$. $\square$

Theorem 197. *[Problem 11-12] Any two points in a connected smooth manifold can be joined by a smooth curve segment.*

Proof. Let M be a connected smooth n-manifold. Define an equivalence relation $\sim$ on M by setting $p \sim q$ if and only if there is a smooth curve segment from p to q. This relation is clearly reflexive and symmetric; if we can show that it is transitive, then the result follows from Theorem 29. Suppose $p \sim q$ and $q \sim r$, let $\gamma : [-1, 0] \to M$ be a smooth curve segment from p to q, and let $\gamma' : [0, 1] \to M$ be a smooth curve segment from q to r. Let (U, φ) be a smooth chart centered at q; by shrinking U, we may assume that $\varphi(U) = \mathbb{B}^n$. Choose $0 < \varepsilon < 1$ so that $\gamma((-\varepsilon, 0]) \subseteq U$ and $\gamma'([0, \varepsilon)) \subseteq U$. Let $\widetilde{\gamma} = \varphi \circ \gamma : (-\varepsilon, 0] \to U$ and $\widetilde{\gamma}' = \varphi \circ \gamma' : [0, \varepsilon) \to U$. Let $\ell : (-\varepsilon, \varepsilon) \to \mathbb{B}^n$ be given by $t \mapsto (t/\varepsilon, 0, \ldots, 0)$. By Proposition 2.25, there is a smooth bump function $\psi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ for $[-\varepsilon/4, \varepsilon/4]$ supported in $(-\varepsilon/2, \varepsilon/2)$. Define $\widetilde{\alpha} : (-\varepsilon, \varepsilon) \to \mathbb{B}^n$ by

$$t \mapsto \psi(t)\ell(t) + (1 - \psi(t))(\widetilde{\gamma}(t) + \widetilde{\gamma}'(t)),$$

taking $\widetilde{\gamma} = 0$ on $(0, \varepsilon)$ and $\widetilde{\gamma}' = 0$ on $(-\varepsilon, 0)$. It is easy to check that $\widetilde{\alpha}$ is smooth on $(-\varepsilon, 0)$, $(-\varepsilon/4, \varepsilon/4)$ and $(0, \varepsilon)$, so $\widetilde{\alpha}$ is smooth. Let $\alpha = \varphi^{-1} \circ \widetilde{\alpha}$; then α, $\gamma|_{[-1,\varepsilon/2)}$, and $\gamma'|_{(\varepsilon/2,1]}$ agree on their overlaps, so we have a smooth curve segment defined on $[-1, 1]$ from p to r. □

Theorem 198. *[Problem 11-13] The length of a smooth curve segment $\gamma : [a, b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral*

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

There is no smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^n)$ with the property that

$$\int_\gamma \omega = L(\gamma)$$

for every smooth curve γ.

Proof. Let $-\gamma$ denote the curve $t \mapsto \gamma(b - t + a)$. Then $L(-\gamma) = L(\gamma)$, but

$$\int_{-\gamma} \omega = -\int_\gamma \omega = -L(\gamma).$$

□

Example 199. [Problem 11-14] Consider the following two covector fields on $\mathbb{R}^3$:

$$\omega = -\frac{4z\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2x\,dz}{x^2+1},$$
$$\eta = -\frac{4xz\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2\,dz}{x^2+1}.$$

(1) Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.
(2) Determine whether either of these covector fields is exact.

(3) For each one that is exact, find a potential function and use it to recompute the line integral.

Let $\gamma : [0,1] \to \mathbb{R}^3$ be given by $t \mapsto (t,t,t)$. Then

$$\begin{aligned}\int_\gamma \omega &= \int_0^1 \left(-\frac{4t}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2t}{t^2+1} \right) dt \\ &= \int_0^1 \frac{4t^3}{(1+t^2)^2}\, dt \\ &= 2\log 2 - 1,\end{aligned}$$

and

$$\begin{aligned}\int_\gamma \eta &= \int_0^1 \left(-\frac{4t^2}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2}{t^2+1} \right) dt \\ &= \int_0^1 \frac{2(t^3 - t^2 + t + 1)}{(1+t^2)^2}\, dt \\ &= \log 2 + 1.\end{aligned}$$

It is easy to check that

$$\frac{\partial}{\partial z}\omega_1 \neq \frac{\partial}{\partial x}\omega_3,$$

so ω is not closed and not exact. It is similarly easy to check that η is closed, so Theorem 11.49 shows that η is exact. Suppose f is a potential for η; it must satisfy

$$\frac{\partial f}{\partial x} = -\frac{4xz}{(x^2+1)^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{y^2+1}, \quad \frac{\partial f}{\partial z} = \frac{2}{x^2+1}.$$

Integrating the first equation, we have

$$f(x,y,z) = \frac{2z}{x^2+1} + C_1(y,z).$$

Then

$$\frac{\partial C_1}{\partial y} = \frac{2y}{y^2+1},$$

so

$$C_1(y,z) = \log(y^2+1) + C_2(z).$$

Finally we have

$$\frac{2}{x^2+1} + \frac{\partial C_2}{\partial z} = \frac{2}{x^2+1},$$

so C_2 is constant. It is now easy to check that for any $C \in \mathbb{R}$,

$$f(x,y,z) = \frac{2z}{x^2+1} + \log(y^2+1) + C$$

is a potential for η. Therefore
$$\int_\gamma \eta = \int_\gamma df = \frac{2}{2} + \log(2) - 0 = \log 2 + 1.$$

Theorem 200. *[Problem 11-15] Let X be a smooth vector field on an open subset $U \subseteq \mathbb{R}^n$. Given a piecewise smooth curve segment $\gamma : [a,b] \to U$, define the **line integral of X over** γ, denoted by $\int_\gamma X \cdot ds$, as*
$$\int_\gamma X \cdot ds = \int_a^b X_{\gamma(t)} \cdot \gamma'(t)\, dt,$$
*where the dot on the right-hand side denotes the Euclidean dot product between tangent vectors at $\gamma(t)$, identified with elements of $\mathbb{R}^n$. A **conservative vector field** is one whose line integral around every piecewise smooth closed curve is zero.*

(1) *X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \operatorname{grad} f$.*
(2) *Suppose $n = 3$. If X is conservative then $\operatorname{curl} X = 0$, where*
$$\operatorname{curl} X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$
(3) *If $U \subseteq \mathbb{R}^3$ is star-shaped, then X is conservative on U if and only if $\operatorname{curl} X = 0$.*

Proof. For any $X \in \mathfrak{X}(U)$, define a smooth covector field ω by $\omega_x(v) = X_x \cdot v$; then
$$\int_\gamma X \cdot ds = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt = \int_\gamma \omega.$$
Part (1) follows immediately from Theorem 11.42, and part (2) follows from Proposition 11.44. Part (3) follows from Theorem 11.49. □

Theorem 201. *[Problem 11-16] Let M be a compact manifold of positive dimension. Every exact covector field on M vanishes at least at two points in each component of M.*

Proof. Let $\omega = df$ be an exact covector field. Let U be a component of M. Since U is compact, f attains a maximum at some $x \in U$ and a minimum at some $y \in U$. If $x = y$ then f is constant on U, so $df = 0$ on U. Otherwise, df vanishes at the distinct points x and y. □

Theorem 202. *[Problem 11-17] Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$ denote the n-torus. For each $j = 1, \ldots, n$, let $\gamma_j : [0,1] \to \mathbb{T}^n$ be the curve segment*
$$\gamma_j(t) = (1, \ldots, e^{2\pi i t}, \ldots, 1).$$
A closed covector field ω on $\mathbb{T}^n$ is exact if and only if $\int_{\gamma_j} \omega = 0$ for $j = 1, \ldots, n$.

Proof. Let $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ be the smooth covering map $\varepsilon^n(x^1, \dots, x^n) = (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$. Note that ε^n is a local diffeomorphism, so Corollary 11.46 shows that $(\varepsilon^n)^*\omega$ is closed and Theorem 11.49 shows that $(\varepsilon^n)^*\omega$ is exact. If ω is exact then $\int_{\gamma_j} \omega = 0$ since each γ_j is a closed curve segment. Conversely, suppose that $\int_{\gamma_j} \omega = 0$ for $j = 1, \dots, n$. Let $\gamma : [0,1] \to \mathbb{T}^n$ be a piecewise smooth closed curve segment and let $\widetilde{\gamma} : [0,1] \to \mathbb{R}^n$ be a lift of γ such that $\gamma = \varepsilon^n \circ \widetilde{\gamma}$; Theorem 41 shows that $\widetilde{\gamma}$ is smooth. Write $\widetilde{\gamma}(0) = (x^1, \dots, x^n)$ and $\widetilde{\gamma}(1) = (x^1 + m_1, \dots, x^n + m_n)$ for integers $m_1, \dots, m_n$. Assume without loss of generality that $x^1 = \cdots = x^n = 0$. For each $i = 1, \dots, n$, let $\alpha_i : [0,1] \to \mathbb{R}^n$ be the straight line path from $(m_1, \dots, m_{i-1}, 0, \dots, 0)$ to $(m_1, \dots, m_i, \dots, 0)$. Let α be the concatenation of $\alpha_1, \dots, \alpha_n$ so that $\alpha(0) = 0$ and $\alpha(1) = (m_1, \dots, m_n)$. Then Theorem 186 shows that

$$\begin{aligned}
\int_\gamma \omega &= \int_{\varepsilon^n \circ \widetilde{\gamma}} \omega \\
&= \int_{\widetilde{\gamma}} (\varepsilon^n)^*\omega \\
&= \int_\alpha (\varepsilon^n)^*\omega \\
&= \int_{\alpha_1} (\varepsilon^n)^*\omega + \cdots + \int_{\alpha_n} (\varepsilon^n)^*\omega, \\
&= \int_{\varepsilon^n \circ \alpha_1} \omega + \cdots + \int_{\varepsilon^n \circ \alpha_n} \omega \\
&= 0
\end{aligned}$$

since each $\varepsilon^n \circ \alpha_i$ is the concatenation of zero or more copies of γ_i. $\square$

Theorem 203. *[Problem 11-18] Suppose C and D are categories, and $\mathcal{F}, \mathcal{G}$ are (covariant or contravariant) functors from C to D. A* ***natural transformation*** *λ from $\mathcal{F}$ to $\mathcal{G}$ is a rule that assigns to each object $X \in \mathrm{Ob}(\mathsf{C})$ a morphism $\lambda_X \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{G}(X))$ in such a way that for every pair of objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\lambda_X"'] & \mathcal{F}(Y) \arrow[d, "\lambda_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(Y).
\end{tikzcd}$$

(If either $\mathcal{F}$ or $\mathcal{G}$ is contravariant, the corresponding horizontal arrow should be reversed.)

(1) *Let* $\mathsf{Vec}_{\mathbb{R}}$ *denote the category of real vector spaces and linear maps, and let* $\mathcal{D}$ *be the contravariant functor from* $\mathsf{Vec}_{\mathbb{R}}$ *to itself that sends each vector space to its dual space and each linear map to its dual map. The assignment* $V \mapsto \xi_V$*, where* $\xi_V : V \to V^{**}$ *is the map defined by* $\xi_V(v)\omega = \omega(v)$*, is a natural transformation from the identity functor of* $\mathsf{Vec}_{\mathbb{R}}$ *to* $\mathcal{D} \circ \mathcal{D}$*.*

(2) *There does not exist a natural transformation from the identity functor of* $\mathsf{Vec}_{\mathbb{R}}$ *to* $\mathcal{D}$*.*

(3) *Let* Diff_1 *be the category of smooth manifolds and diffeomorphisms and* VB *the category of smooth vector bundles and smooth bundle homomorphisms, and let* $T, T^* : \mathsf{Diff}_1 \to \mathsf{VB}$ *be the tangent and cotangent functors, respectively. There does not exist a natural transformation from* T *to* T^**.*

(4) *Let* $\mathfrak{X} : \mathsf{Diff}_1 \to \mathsf{Vec}_{\mathbb{R}}$ *be the covariant functor given by* $M \mapsto \mathfrak{X}(M)$*,* $F \mapsto F_*$*; and let* $\mathfrak{X} \times \mathfrak{X} : \mathsf{Diff}_1 \to \mathsf{Vec}_{\mathbb{R}}$ *be the covariant functor given by* $M \mapsto \mathfrak{X}(M) \times \mathfrak{X}(M)$*,* $F \mapsto F_* \times F_*$*. The Lie bracket is a natural transformation from* $\mathfrak{X} \times \mathfrak{X}$ *to* $\mathfrak{X}$*.*

Proof. Theorem 187 proves (1) and (2). If there is a natural transformation from T to T^*, then taking the one-point space $\{*\}$ produces a natural transformation from the identity functor of $\mathsf{Vec}_{\mathbb{R}}$ to $\mathcal{D}$, contradicting (2). This proves (3). Corollary 8.31 proves (4). $\square$

Chapter 12. Tensors

Theorem 204. *[Exercise 12.18]* T^kT^*M*,* T^kTM*, and* $T^{(k,l)}TM$ *have natural structures as smooth vector bundles over* M*.*

Proof. Let $n = \dim M$. Let (U, φ) be a smooth chart for M with coordinate functions (x^i). Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n(k+l)}$ by

$$\Phi\left(a^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} \left.\frac{\partial}{\partial x^{i_1}}\right|_p \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_k}}\right|_p \otimes dx^{j_1}\big|_p \otimes \cdots \otimes dx^{j_l}\big|_p \right)$$
$$= (p, (a^{1,\ldots,1}_{1,\ldots,1}, \ldots, a^{n,\ldots,n}_{n\ldots,n})).$$

Suppose $(\widetilde{U}, \widetilde{\varphi})$ is another smooth chart with coordinate functions $(\widetilde{x}^i)$, and let $\widetilde{\Phi} : \pi^{-1}(\widetilde{U}) \to \widetilde{U} \times \mathbb{R}^n$ be defined analogously. It follows from (11.6) and (11.7) that

$$\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p = \frac{\partial x^i}{\partial \widetilde{x}^j}(p) \left.\frac{\partial}{\partial x^i}\right|_p, \quad d\widetilde{x}^j|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, dx^i|_p,$$

where $\widetilde{x} = \widetilde{\varphi} \circ \varphi^{-1}$ and $x = \varphi \circ \widetilde{\varphi}^{-1} = \widetilde{x}^{-1}$, and p denotes either a point in M or its coordinate representation as appropriate. Therefore on $\pi^{-1}(U \cap \widetilde{U})$ we have

$$\begin{aligned}
&(\Phi \circ \widetilde{\Phi}^{-1})(p, (\widetilde{a}^{1,\dots,1}_{1,\dots,1}, \dots, \widetilde{a}^{n,\dots,n}_{n\dots,n})) \\
&= \Bigg(p, \Bigg(\frac{\partial x^1}{\partial \widetilde{x}^{i_1}}(p) \cdots \frac{\partial x^1}{\partial \widetilde{x}^{i_k}}(p) \frac{\partial \widetilde{x}^{j_1}}{\partial x^1}(p) \cdots \frac{\partial \widetilde{x}^{j_l}}{\partial x^1}(p) \widetilde{a}^{i_1,\dots,i_k}_{j_1,\dots,j_l}, \\
&\qquad \dots, \frac{\partial x^n}{\partial \widetilde{x}^{i_1}}(p) \cdots \frac{\partial x^n}{\partial \widetilde{x}^{i_k}}(p) \frac{\partial \widetilde{x}^{j_1}}{\partial x^n}(p) \cdots \frac{\partial \widetilde{x}^{j_l}}{\partial x^n}(p) \widetilde{a}^{i_1,\dots,i_k}_{j_1,\dots,j_l}\Bigg)\Bigg).
\end{aligned}$$

It follows from the vector bundle chart lemma that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are smooth local trivializations. □

Theorem 205. *[Proposition 12.25] Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.*

(1) $F^*(fB) = (f \circ F)F^*B$.
(2) $F^*(A \otimes B) = F^*A \otimes F^*B$.
(3) $F^*(A + B) = F^*A + F^*B$.
(4) *F^*B is a (continuous) tensor field, and is smooth if B is smooth.*
(5) $(G \circ F)^*B = F^*(G^*B)$.
(6) $(\mathrm{Id}_N)^*B = B$.

Proof. For (1),

$$(F^*(fB))_p = dF^*_p((fB)_{F(p)}) = dF^*_p(f(F(p))B_{F(p)}) = (f \circ F)(p)(F^*B)_p$$

since dF^*_p is linear. For (2),

$$\begin{aligned}
(F^*(A \otimes B))_p(v_1, \dots, v_{k+l}) &= (A \otimes B)_p(dF_p(v_1), \dots, dF_p(v_{k+l})) \\
&= A_p(dF_p(v_1), \dots, dF_p(v_k))B_p(dF_p(v_{k+1}), \dots, dF_p(v_{k+l})) \\
&= (F^*A)_p(v_1, \dots, v_k)(F^*B)_p(v_{k+1}, \dots, v_{k+l}).
\end{aligned}$$

Part (3) follows from the fact that dF^*_p is linear. For (4), let $p \in M$ be arbitrary, and choose smooth coordinates (y^j) for N in a neighborhood V of $F(p)$. Let $U = F^{-1}(V)$, which is a neighborhood of p. Writing B in coordinates as

$$B = B_{j_1,\dots,j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k}$$

for continuous functions $B_{j_1,\dots,j_k}$ on V, we have

$$\begin{aligned}
F^*B &= F^*(B_{j_1,\dots,j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k}) \\
&= (B_{j_1,\dots,j_k} \circ F)F^*(dy^{j_1} \otimes \cdots \otimes dy^{j_k}) \\
&= (B_{j_1,\dots,j_k} \circ F)F^*dy^{j_1} \otimes \cdots \otimes F^*dy^{j_k} \\
&= (B_{j_1,\dots,j_k} \circ F)d(y^{j_1} \circ F) \otimes \cdots \otimes d(y^{j_k} \circ F),
\end{aligned}$$

which is continuous, and is smooth if B is smooth. Part (5) follows from the fact that

$$\begin{aligned} d(G\circ F)_p^*(\alpha)(v_1,\dots,v_k) &= \alpha(dG_{F(p)}(dF_p(v_1)),\dots,dG_{F(p)}(dF_p(v_k))) \\ &= dG_{F(p)}^*(\alpha)(dF_p(v_1),\dots,dF_p(v_k)) \\ &= dF_p^*(dG_{F(p)}^*(\alpha))(v_1,\dots,v_k). \end{aligned}$$

Part (6) is obvious. □

Example 206. [Problem 12-1] Give an example of finite-dimensional vector spaces V and W and a specific element $\alpha \in V\otimes W$ that cannot be expressed as $v\otimes w$ for $v\in V$ and $w\in W$.

Take $V = W = \mathbb{R}^2$, let $e_1 = (1,0)$ and $e_2 = (0,1)$, and let $\alpha = e_1\otimes e_1 + e_2\otimes e_2$. Suppose $\alpha = v\otimes w$ for $v, w\in\mathbb{R}^2$, and write $v = v^i e_i$ and $w = w^j e_j$. Then $\alpha = v^i w^j e_i\otimes e_j$, so

$$v^1w^1 = 1,\quad v^1w^2 = 0,\quad v^2w^1 = 0,\quad v^2w^2 = 1.$$

This implies that $w^1\neq 0$, so $v^2 = 0$, contradicting $v^2w^2 = 1$.

Theorem 207. *[Problem 12-2] For any finite-dimensional real vector space V, there are canonical isomorphisms $\mathbb{R}\otimes V\cong V\cong V\otimes\mathbb{R}$.*

Proof. We can define a multilinear map $f:\mathbb{R}\times V\to V$ by $(r,v)\mapsto rv$, which induces a linear map $\widetilde{f}:\mathbb{R}\otimes V\to V$ satisfying $\widetilde{f}(r\otimes v) = rv$. It is easy to check that the linear map $v\mapsto 1\otimes v$ is an inverse to $\widetilde{f}$, so $\widetilde{f}$ is an isomorphism. A similar argument shows that $V\cong V\otimes\mathbb{R}$. □

Theorem 208. *[Problem 12-3] Let V and W be finite-dimensional real vector spaces. The tensor product space $V\otimes W$ is uniquely determined up to canonical isomorphism by its characteristic property. More precisely, suppose $\widetilde{\pi}: V\times W\to Z$ is a bilinear map into a vector space Z with the following property: for any bilinear map $B: V\times W\to Y$, there is a unique linear map $\widetilde{B}: Z\to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} V\times W \arrow[r, "B"] \arrow[d, "\widetilde{\pi}"'] & Y. \\ Z \arrow[ur, dashed, "\widetilde{B}"'] & \end{tikzcd}$$

Then there is a unique isomorphism $\Phi: V\otimes W\to Z$ such that $\widetilde{\pi} = \Phi\circ\pi$, where $\pi: V\times W\to V\otimes W$ is the canonical projection.

Proof. Since $\widetilde{\pi}$ is bilinear, there exists a unique linear map $\Phi: V\otimes W\to Z$ such that $\widetilde{\pi} = \Phi\circ\pi$. Similarly, there exists a unique linear map $\Psi: Z\to V\otimes W$ such that

$\pi = \Psi \circ \widetilde{\pi}$. We have $\Phi \circ \Psi \circ \widetilde{\pi} = \Phi \circ \pi = \widetilde{\pi}$, so $\Phi \circ \Psi = \mathrm{Id}_Z$ by uniqueness. Similarly, $\Psi \circ \Phi \circ \pi = \Psi \circ \widetilde{\pi} = \pi$ implies that $\Psi \circ \Phi = \mathrm{Id}_{V \otimes W}$ by uniqueness. Therefore Φ is an isomorphism. $\square$

Theorem 209. *[Problem 12-4] Let $V_1, \ldots, V_k$ and W be finite-dimensional real vector spaces. There is a canonical isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \otimes W \cong L(V_1, \ldots, V_k; W).$$

Proof. Proposition 12.10 shows that

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}),$$

so it remains to show that

$$L(V_1, \ldots, V_k; \mathbb{R}) \otimes W \cong L(V_1, \ldots, V_k; W).$$

The bilinear map $\varphi : L(V_1, \ldots, V_k; \mathbb{R}) \times W \to L(V_1, \ldots, V_k; W)$ defined by

$$\varphi(f, w)(v_1, \ldots, v_k) = f(v_1, \ldots, v_k)w$$

induces a linear map $\widetilde{\varphi} : L(V_1, \ldots, V_k; \mathbb{R}) \otimes W \to L(V_1, \ldots, V_k; W)$ satisfying

$$\widetilde{\varphi}(f \otimes w)(v_1, \ldots, v_k) = f(v_1, \ldots, v_k)w.$$

It is easy to check that $\widetilde{\varphi}$ takes a basis of $L(V_1, \ldots, V_k; \mathbb{R}) \otimes W$ to a basis of $L(V_1, \ldots, V_k; W)$, so $\widetilde{\varphi}$ is an isomorphism. $\square$

Theorem 210. *[Problem 12-5] Let V be an n-dimensional real vector space. Then*

$$\dim \Sigma^k(V^*) = \binom{n+k-1}{k}.$$

Proof. Let $\{\varepsilon^1, \ldots, \varepsilon^n\}$ be a basis for V^*; we want to show that

$$\mathcal{B} = \left\{\varepsilon^{i_1} \cdots \varepsilon^{i_k} : 1 \le i_1 \le \cdots \le i_k \le n\right\}$$

is a basis for $\Sigma^k(V^*)$, from which the result follows immediately. It is clear that $\mathcal{B}$ is linearly independent. Let $\alpha \in \Sigma^k(V^*)$ and write

$$\alpha = a_{i_1, \ldots, i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

We have

$$\begin{aligned} \alpha = \operatorname{Sym} \alpha &= a_{i_1, \ldots, i_k} \operatorname{Sym} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \\ &= a_{i_1, \ldots, i_k} \varepsilon^{i_1} \cdots \varepsilon^{i_k}, \end{aligned}$$

which is in the span of $\mathcal{B}$ after reordering the indices $i_1, \ldots, i_k$ in each term $\varepsilon^{i_1} \cdots \varepsilon^{i_k}$ to be in ascending order. This shows that $\mathcal{B}$ is a basis for $\Sigma^k(V^*)$. $\square$

Theorem 211. *[Problem 12-6]*

(1) *Let α be a covariant k-tensor on a finite-dimensional real vector space V. Then* $\operatorname{Sym}\alpha$ *is the unique symmetric k-tensor satisfying*
$$(\operatorname{Sym}\alpha)(v,\dots,v)=\alpha(v,\dots,v)$$
for all $v\in V$.

(2) *The symmetric product is associative: for all symmetric tensors α,β,γ,*
$$(\alpha\beta)\gamma=\alpha(\beta\gamma).$$

(3) *Let $\omega^1,\dots,\omega^k$ be covectors on a finite-dimensional vector space. Their symmetric product satisfies*
$$\omega^1\cdots\omega^k=\frac{1}{k!}\sum_{\sigma\in S_k}\omega^{\sigma(1)}\otimes\cdots\otimes\omega^{\sigma(k)}.$$

Proof. Let $\{E_1,\dots,E_k\}$ be a basis for V. For (1),
$$(\operatorname{Sym}\alpha)(v,\dots,v)=\frac{1}{k!}\sum_{\sigma\in S_k}\alpha(v,\dots,v)=\alpha(v,\dots,v).$$
The uniqueness of $\operatorname{Sym}\alpha$ follows from the fact that
$$\left\{(E_{i_1}+\cdots+E_{i_n})^{\otimes k}:1\le n\le k,\ 1\le i_1<\cdots<i_n\le k\right\}$$
spans $V^{\otimes k}$. For (2), we have
$$\begin{aligned}(\operatorname{Sym}(\operatorname{Sym}\alpha\otimes\beta)\otimes\gamma)(v,\dots,v)&=(\operatorname{Sym}\alpha\otimes\beta)(v,\dots,v)\gamma(v,\dots,v)\\&=\alpha(v,\dots,v)\beta(v,\dots,v)\gamma(v,\dots,v)\\&=\alpha(v,\dots,v)(\operatorname{Sym}\beta\otimes\gamma)(v,\dots,v)\\&=(\operatorname{Sym}\alpha\otimes(\operatorname{Sym}\beta\otimes\gamma))(v,\dots,v),\end{aligned}$$
so the result follows from (1). (Note that the tuples $(v,\dots,v)$ may have different numbers of elements.) Part (3) is obvious. □

Example 212. [Problem 12-7] Let (e^1,e^2,e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. The tensor $e^1\otimes e^2\otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

Let (e_1,e_2,e_3) be the standard basis for $\mathbb{R}^3$. Suppose that
$$e^1\otimes e^2\otimes e^3=\alpha+\beta$$
where α is alternating and β is symmetric. We have
$$1=(e^1\otimes e^2\otimes e^3)(e_1,e_2,e_3)=\alpha(e_1,e_2,e_3)+\beta(e_1,e_2,e_3)$$
but
$$\begin{aligned}0=(e^1\otimes e^2\otimes e^3)(e_2,e_3,e_1)&=\alpha(e_2,e_3,e_1)+\beta(e_2,e_3,e_1)\\&=\alpha(e_1,e_2,e_3)+\beta(e_1,e_2,e_3),\end{aligned}$$

which is a contradiction.

Chapter 13. Riemannian Metrics

Theorem 213. *[Exercise 13.21] Let (M, g) be a Riemannian n-manifold with or without boundary. For any immersed k-dimensional submanifold $S \subseteq M$ with or without boundary, the normal bundle NS is a smooth rank-$(n-k)$ subbundle of $TM|_S$. For each $p \in S$, there is a smooth frame for NS on a neighborhood of p that is orthonormal with respect to g.*

Proof. Let $p \in M$ be arbitrary, and let $(X_1, \dots, X_k)$ be a smooth local frame for TS over some neighborhood V of p in M. Because immersed submanifolds are locally embedded, by shrinking V if necessary, we may assume that it is a single slice in some coordinate ball or half-ball $U \subseteq \mathbb{R}^n$. Since V is closed in U, Theorem 122(c) shows that we can complete $(X_1, \dots, X_k)$ to a smooth local frame $(\widetilde{X}_1, \dots, \widetilde{X}_n)$ for $TM|_S$ over U, and then Proposition 13.6 yields a smooth orthonormal frame (E_j) over U such that $\operatorname{span}(E_1|_p, \dots, E_k|_p) = \operatorname{span}(X_1|_p, \dots, X_k|_p) = T_pS$ for each $p \in U$. It follows that $(E_{k+1}, \dots, E_n)$ restricts to a smooth orthonormal frame for NS over V. Thus NS satisfies the local frame criterion in Lemma 10.32, and is therefore a smooth subbundle of $TM|_S$. $\square$

Theorem 214. *[Exercise 13.24] Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Then $L_{\widetilde{g}}(F \circ \gamma) = L_g(\gamma)$ for every piecewise smooth curve segment γ in M.*

Proof. We have

$$\begin{aligned}
L_{\widetilde{g}}(F \circ \gamma) &= \int_a^b \left|(F \circ \gamma)'(t)\right|_{\widetilde{g}} \, dt \\
&= \int_a^b \left|dF_{\gamma(t)}(\gamma'(t))\right|_{\widetilde{g}} \, dt \\
&= \int_a^b \sqrt{\widetilde{g}_{(F\circ\gamma)(t)}(dF_{\gamma(t)}(\gamma'(t)), dF_{\gamma(t)}(\gamma'(t)))} \, dt \\
&= \int_a^b \sqrt{(F^*\widetilde{g})(\gamma'(t), \gamma'(t))} \, dt \\
&= \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \, dt \\
&= L_g(\gamma).
\end{aligned}$$

$\square$

Theorem 215. *[Exercise 13.27] Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds and $F : M \to \widetilde{M}$ is a Riemannian isometry. Then*

$$d_{\widetilde{g}}(F(p), F(q)) = d_g(p, q)$$

for all $p, q \in M$.

Proof. If γ is a piecewise smooth curve segment from p to q, then $F \circ \gamma$ is a piecewise smooth curve segment from $F(p)$ to $F(q)$, and Theorem 214 shows that $L_g(\gamma) = L_{\widetilde{g}}(F \circ \gamma)$. Similarly, if $\widetilde{\gamma}$ is a piecewise smooth curve segment from $F(p)$ to $F(q)$ then $F^{-1} \circ \widetilde{\gamma}$ is a piecewise smooth curve segment from p to q and $L_{\widetilde{g}}(\widetilde{\gamma}) = L_g(F^{-1} \circ \widetilde{\gamma})$. □

Theorem 216. *[Problem 13-1] If (M, g) is a Riemannian n-manifold with or without boundary, let $UM \subseteq TM$ be the subset $UM = \left\{(x, v) \in TM : |v|_g = 1\right\}$, called the **unit tangent bundle of** M. Then UM is a smooth fiber bundle over M with model fiber $\mathbb{S}^{n-1}$.*

Proof. Define $\ell : TM \to \mathbb{R}$ by $(x, v) \mapsto g(x)(v, v)$ and for fixed $x \in M$, define $\ell_x : T_xM \to \mathbb{R}$ by $v \mapsto g(x)(v, v)$. In any smooth local coordinates around x we have $D\ell(v)(u) = 2g(x)(v, u)$ and in particular $D\ell(v)(v) = 2g(x)(v, v) > 0$ for all $v \neq 0$. Therefore Corollary 5.14 shows that $UM = \ell^{-1}(\{1\})$ is a properly embedded submanifold of TM. Let $\pi : TM \to M$ be the canonical projection. Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a smooth local trivialization. Define

$$\begin{aligned} \varphi : \pi^{-1}(U) \cap UM &\to U \times \mathbb{S}^{n-1} \\ (x, v) &\mapsto \left(x, \frac{(\pi_{\mathbb{R}^n} \circ \Phi)(x, v)}{\|(\pi_{\mathbb{R}^n} \circ \Phi)(x, v)\|}\right) \end{aligned}$$

and

$$\begin{aligned} \psi : U \times \mathbb{S}^{n-1} &\to \pi^{-1}(U) \cap UM \\ (x, v) &\mapsto \frac{\Phi^{-1}(x, v)}{|\Phi^{-1}(x, v)|_g}. \end{aligned}$$

Then φ and ψ are smooth and inverses of each other, so φ is a local trivialization of UM over M. □

Theorem 217. *[Problem 13-2] Suppose that E is a smooth vector bundle over a smooth manifold M with or without boundary, and $V \subseteq E$ is an open subset with the property that for each $p \in M$, the intersection of V with the fiber E_p is convex and nonempty. By a "section of V", we mean a (local or global) section of E whose image lies in V.*

(1) *There exists a smooth global section of V.*

(2) *Suppose* $\sigma : A \to V$ *is a smooth section of* V *defined on a closed subset* $A \subseteq M$. *(This means that* σ *extends to a smooth section of* V *in a neighborhood of each point of* A.*) There exists a smooth global section* $\widetilde{\sigma}$ *of* V *whose restriction to* A *is equal to* σ. *If* V *contains the image of the zero section of* E, *then* $\widetilde{\sigma}$ *can be chosen to be supported in any predetermined neighborhood of* A.

Proof. For (1), let $p \in M$ and choose a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $p \in U$. Choose any $v \in \pi^{-1}(\{p\})$ and choose neighborhoods U_p of p and F_p of $v' = (\pi_{\mathbb{R}^n} \circ \Phi)(v)$ such that $U_p \times F_p \subseteq \Phi(\pi^{-1}(U) \cap V)$. Define $\tau_p : U_p \to V$ by $x \mapsto \Phi^{-1}(x, v')$. The family of sets $\{U_p : p \in M\}$ is an open cover of M. Let $\{\psi_p : p \in M\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq U_p$. For each $p \in M$, the product $\psi_p \tau_p$ is smooth on W_p by Theorem 159, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \setminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\tau : M \to V$ by

$$\tau(x) = \sum_{p \in M} \psi_p(x) \tau_p(x).$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. The fact that $\sum_{p \in M} \psi_p = 1$ implies that $\tau(x)$ is a convex combination of vectors in E_x for every $x \in M$, so the image of τ does indeed lie in V.

For (2), let $\tau : M \to V$ be a smooth global section of V. If V contains the imaage of the zero section of E, then let U be some neighborhood of A and replace τ with the zero section of E. For each $p \in A$, choose a neighborhood W_p of p and a local section $\widetilde{\sigma}_p : W_p \to V$ of V that agrees with σ on $W_p \cap A$. If U was chosen, then replacing W_p by $W_p \cap U$ we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{\sigma}_p$ is smooth on W_p and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. Thus we can define $\widetilde{\sigma} : M \to V$ by

$$\widetilde{\sigma}(x) = \psi_0(x)\tau(x) + \sum_{p \in A} \psi_p(x) \widetilde{\sigma}_p(x).$$

If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{\sigma}_p(x) = \sigma(x)$ for each p such that $\psi_p(x) \neq 0$, so

$$\widetilde{\sigma}(x) = \sum_{p \in A} \psi_p(x)\sigma(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) \sigma(x) = \sigma(x),$$

so $\widetilde{\sigma}$ is indeed an extension of σ. The fact that $\psi_0 + \sum_{p \in A} \psi_p = 1$ implies that $\widetilde{\sigma}(x)$ is a convex combination of vectors in E_x for every $x \in M$, so the image of $\widetilde{\sigma}$ does indeed lie in V. If U was chosen, then it follows from Lemma 1.13(b) that

$$\operatorname{supp}\widetilde{\sigma} = \overline{\bigcup_{p\in A}\operatorname{supp}\psi_p} = \bigcup_{p\in A}\operatorname{supp}\psi_p \subseteq U.$$

□

Theorem 218. *Suppose (M,g) is a Riemannian manifold, and X is a vector field on M. Given any positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth vector field $\widetilde{X} \in \mathfrak{X}(M)$ that is δ-close to X. If X is smooth on a closed subset $A \subseteq M$, then $\widetilde{X}$ can be chosen to be equal to X on A. (Two vector fields $X, \widetilde{X}$ are δ-close if $\left|X_x - \widetilde{X}_x\right|_g < \delta(x)$ for all $x \in M$.*

Proof. If X is smooth on the closed subset A, then by Lemma 8.6, there is a smooth vector field $X_0 \in \mathfrak{X}(M)$ that agrees with X on A. Let

$$U_0 = \{y \in M : |X_0|_y - X_y|_g < \delta(y)\}.$$

Then U_0 is an open subset containing A. (If there is no such set A, we just take $U_0 = A = \emptyset$ and $F_0 = 0$.)

We will show that there are countably many points $\{x_i\}_{i=1}^{\infty}$ in $M \setminus A$ and smooth coordinate balls U_i of x_i in $M \setminus A$ such that $\{U_i\}_{i=1}^{\infty}$ is an open cover of $M \setminus A$ and

$$(*) \qquad |X_y - X_i|_y|_g < \delta(y)$$

for all $y \in U_i$, where $X_i \in \mathfrak{X}(U_i)$ is a constant coefficient vector field such that $X_i|_{x_i} = X_{x_i}$. To see this, for any $x \in M \setminus A$, let U_x be a smooth coordinate ball around x contained in $M \setminus A$ and small enough that

$$\delta(y) > \frac{1}{2}\delta(x) \quad \text{and} \quad \left|X_y - X^{(x)}|_y\right|_g < \frac{1}{2}\delta(x)$$

for all $y \in U_x$, where $X^{(x)} \in \mathfrak{X}(U_x)$ is a constant coefficient vector field such that $X^{(x)}|_x = X_x$. (Such a neighborhood exists by continuity of δ and X.) Then if $y \in U_x$, we have

$$\left|X_y - X^{(x)}|_y\right|_g < \frac{1}{2}\delta(x) < \delta(y).$$

The collection $\{U_x : x \in M \setminus A\}$ is an open cover of $M \setminus A$. Choosing a countable subcover $\{U_{x_i}\}_{i=1}^{\infty}$ and setting $U_i = U_{x_i}$ and $X_i = X^{(x_i)}$, we have (*).

Let $\{\varphi_0, \varphi_i\}$ be a smooth partition of unity subordinate to the cover $\{U_0, U_i\}$ of M, and define $\widetilde{X} \in \mathfrak{X}(M)$ by

$$\widetilde{X}_y = \varphi_0(y) X_0|_y + \sum_{i\geq 1}\varphi_i(y) X_i|_y.$$

Then clearly $\widetilde{X}$ is smooth, and is equal to X on A. For any $y \in M$, the fact that $\sum_{i\geq 0}\varphi_i = 1$ implies that

$$\begin{aligned}\left|\widetilde{X}_y - X_y\right|_g &= \left|\varphi_0(y)X_0|_y + \sum_{i\geq 1}\varphi_i(y)X_i|_y - \left(\varphi_0(y) + \sum_{i\geq 1}\varphi_i(y)\right)X_y\right|_g \\ &\leq \varphi_0(y)\left|X_0|_y - X_y\right|_g + \sum_{i\geq 1}\varphi_i(y)\left|X_i|_y - X_y\right|_g \\ &< \varphi_0(y)\delta(y) + \sum_{i\geq 1}\varphi_i(y)\delta(y) = \delta(y).\end{aligned}$$

□

Lemma 219. *Let V be a real inner product space and let $\|\cdot\|$ be the norm on V. Let $\{u_1, \ldots, u_k\}$ be an orthonormal set of vectors in V with $\|u_i\| = 1$ for $i = 1, \ldots, k$. Let $v \in V$ with $\|v\| = 1$.*

(1) *We have*
$$\langle u_1, v\rangle^2 + \cdots + \langle u_k, v\rangle^2 \leq 1,$$
and equality holds if and only if the set $\{v, u_1, \ldots, u_k\}$ is linearly dependent.

(2) *If $w \in V$ and*
$$\|v - w\|^2 < 1 - \langle u_1, v\rangle^2 - \cdots - \langle u_k, v\rangle^2,$$
then $\{w, u_1, \ldots, u_k\}$ are linearly independent.

Proof. For (1), complete $\{u_1, \ldots, u_k\}$ to an orthonormal basis $\{u_1, \ldots, u_n\}$ of V. Write $v = a_1u_1 + \cdots + a_nu_n$ where $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$\begin{aligned}\langle u_1, v\rangle^2 + \cdots + \langle u_k, v\rangle^2 &= \langle u_1, a_1u_1\rangle^2 + \cdots + \langle u_k, a_ku_k\rangle^2 \\ &= a_1^2 + \cdots + a_k^2 \\ &= 1 - a_{k+1}^2 - \cdots - a_n^2\end{aligned}$$

since $\|v\| = 1$. Clearly the last expression is equal to 1 if and only if $a_{k+1}, \ldots, a_n = 0$, which is true if and only if v is in the span of $\{u_1, \ldots, u_k\}$. For (2), suppose $w = a_1u_1 + \cdots + a_ku_k$ where $a_1, \ldots, a_k \in \mathbb{R}$. Then

$$\begin{aligned}0 &> \left\|v - \sum_{i=1}^k a_iu_i\right\|^2 - 1 + \sum_{i=1}^k \langle u_i, v\rangle^2 \\ &= 1 - 2\left\langle v, \sum_{i=1}^k a_iu_i\right\rangle + \left\|\sum_{i=1}^k a_iu_i\right\|^2 - 1 + \sum_{i=1}^k \langle u_i, v\rangle^2\end{aligned}$$

$$= \sum_{i=1}^{k} (a_i^2 - 2a_i \langle u_i, v\rangle + \langle u_i, v\rangle^2)$$
$$= \sum_{i=1}^{k} (a_i - \langle u_i, v\rangle)^2,$$

which is a contradiction. □

Theorem 220. *[Problem 13-3] Let M be a smooth manifold.*

(1) *If there exists a global nonvanishing vector field on M, then there exists a global* smooth *nonvanishing vector field.*
(2) *If there exists a linearly independent k-tuple of vector fields on M, then there exists such a k-tuple of* smooth *vector fields.*

Proof. Let g be a Riemannian metric on M. For (1), let X be a global nonvanishing vector field on M. By Theorem 218, there is a smooth vector field $\widetilde{X} \in \mathfrak{X}(M)$ such that

$$|X_x - \widetilde{X}_x|_g < |X_x|_g$$

for all $x \in M$. This implies that $|\widetilde{X}_x|_g > 0$ for all $x \in M$, so $\widetilde{X}$ is nonvanishing. For (2), let $(X_1, \ldots, X_k)$ be a linearly independent k-tuple of vector fields on M. Using the Gram-Schmidt process, we may assume that $(X_1, \ldots, X_k)$ is an orthonormal k-tuple of vector fields. Assume that $X_1, \ldots, X_{j-1}$ are already smooth. Define

$$\delta(x) = \sqrt{1 - \sum_{i \neq j} \langle X_j|_x, X_i|_x\rangle_g^2};$$

then $\delta : M \to \mathbb{R}$ is a positive continuous function by Lemma 219. By Theorem 218, there is a smooth vector field $\widetilde{X}_j \in \mathfrak{X}(M)$ such that

$$|(X_j|_x - \widetilde{X}_j|_x)|_g^2 < 1 - \sum_{i \neq j} \langle X_j|_x, X_i|_x\rangle_g^2$$

for all $x \in M$. Then Lemma 219 shows that $(X_1, \ldots, \widetilde{X}_j, \ldots, X_k)$ is linearly independent. Apply the Gram-Schmidt process to $(X_1, \ldots, \widetilde{X}_j)$ to obtain an orthonormal k-tuple of vector fields such that the first j vector fields are smooth. Repeating this procedure k times produces an orthonormal k-tuple of smooth vector fields. □

Example 221. [Problem 13-4] Let $\mathring{g}$ denote the round metric on $\mathbb{S}^n$. Compute the coordinate representation of $\mathring{g}$ in stereographic coordinates.

We have

$$(x^1, \ldots, x^{n+1}) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1},$$

so
$$dx^j = \frac{2(|u|^2 - 2u^j + 1)}{(|u|^2+1)^2}du^j - \sum_{i\neq j}\frac{4u^iu^j}{(|u|^2+1)^2}du^i$$
for each $j = 1, \ldots, n$ and
$$dx^{n+1} = \sum_{i=1}^{n} \frac{4u^i}{(|u|^2+1)^2}du^i.$$
Therefore
$$\begin{aligned}\mathring{g} &= (dx^1)^2 + \cdots + (dx^{n+1})^2 \\ &= \frac{4}{(|u|^2+1)^2}((du^1)^2 + \cdots + (du^n)^2).\end{aligned}$$

Theorem 222. *[Problem 13-5] Suppose (M, g) is a Riemannian manifold. A smooth curve $\gamma : J \to M$ is said to be a* ***unit-speed curve*** *if $|\gamma'(t)|_g = 1$. Every smooth curve with nowhere-vanishing velocity has a unit-speed reparametrization.*

Proof. We can assume that J is open. Choose any $t_0 \in J$ and define $\ell : J \to \mathbb{R}$ by
$$\ell(t) = \int_{t_0}^{t} |\gamma'(x)|_g \, dx.$$
Then $x \mapsto |\gamma'(x)|_g$ is smooth since $v \mapsto |v|_g$ is smooth on $\{(x, v) \in TM : v \neq 0\}$ and $\gamma'(x) \neq 0$ for all $x \in J$. Therefore ℓ is smooth, and since $\ell'(t) \neq 0$ for all $t \in J$, the inverse function theorem shows that $\ell^{-1} : \ell(J) \to J$ is smooth. Define $\gamma_1 : \ell(J) \to M$ by $\gamma_1 = \gamma \circ \ell^{-1}$. We have
$$\begin{aligned}\gamma_1'(x) &= (\ell^{-1})'(x)\gamma'(\ell^{-1}(x)) \\ &= \frac{\gamma'(\ell^{-1}(x))}{\ell'(\ell^{-1}(x))} \\ &= \frac{\gamma'(\ell^{-1}(x))}{|\gamma'(\ell^{-1}(x))|_g},\end{aligned}$$
so γ_1 is a unit-speed reparametrization of γ. □

Theorem 223. *[Problem 13-6] Every Riemannian 1-manifold is flat.*

Proof. Let (M, g) be a Riemannian 1-manifold. Suppose $x \in M$ and (U, φ) is a smooth chart containing x. By shrinking U, we can assume that U is connected. Therefore $\varphi^{-1} : \varphi(U) \to U$ is a smooth curve, and Theorem 222 shows that there is a diffeomorphism $\psi : E \to \varphi(U)$ such that $\varphi^{-1} \circ \psi$ is a unit-speed curve. We want to show that $\varphi^{-1} \circ \psi :$

$E \to U$ is an isometry. If $v, w \in T_pE$ are nonzero vectors then $v = r\, d/dt|_p$ and $w = s\, d/dt|_p$ for some $r, s \in \mathbb{R}$, so

$$\begin{aligned}
((\varphi^{-1} \circ \psi)^* g_p)(v, w) &= rs((\varphi^{-1} \circ \psi)^* g_p)\left(\left.\frac{d}{dt}\right|_p, \left.\frac{d}{dt}\right|_p\right) \\
&= rs \left| d(\varphi^{-1} \circ \psi)_p \left(\left.\frac{d}{dt}\right|_p\right)\right|_g^2 \\
&= rs \left|(\varphi^{-1} \circ \psi)'(p)\right|_g^2 \\
&= rs \\
&= \overline{g}_{(\varphi^{-1} \circ \psi)(p)}(v, w).
\end{aligned}$$

□

Theorem 224. *[Problem 13-7] A product of flat metrics is flat.*

Proof. If $(R_1, g_1), \ldots, (R_k, g_k)$ are Riemannian manifolds, then

$$(R_1 \times \cdots \times R_k, g_1 \oplus \cdots \oplus g_k)$$

is also a Riemannian manifold. Let $(x_1, \ldots, x_k) \in R_1 \times \cdots \times R_k$. For each $i = 1, \ldots, k$, let $F_i : U_i \to V_i$ be an isometry such that $x_i \in U_i$. Then $F_1 \times \cdots \times F_k$ is an isometry from the neighborhood $U_1 \times \cdots \times U_k$ to the open set $V_1 \times \cdots \times V_k$, since

$$\begin{aligned}
(F_1 \times \cdots \times F_k)^*(\overline{g} \oplus \cdots \oplus \overline{g}) &= F_1^* \overline{g} \oplus \cdots \oplus F_k^* \overline{g} \\
&= g_1 \oplus \cdots \oplus g_k.
\end{aligned}$$

□

Corollary 225. *[Problem 13-8] Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$, and let g be the metric on $\mathbb{T}^n$ induced from the Euclidean metric on $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$). Then g is flat.*

Theorem 226. *[Problem 13-10] The shortest path between two points in Euclidean space is a straight line segment. More precisely, for $x, y \in \mathbb{R}^n$, let $\gamma : [0, 1] \to \mathbb{R}^n$ be the curve segment $\gamma(t) = x + t(y - x)$. Any other piecewise smooth curve segment $\widetilde{\gamma}$ from x to y satisfies $L_{\overline{g}}(\widetilde{\gamma}) > L_{\overline{g}}(\gamma)$ unless $\widetilde{\gamma}$ is a reparametrization of γ.*

Proof. First note that if $\widetilde{\gamma} : [a, b] \to \mathbb{R}^n$ then

$$\int_a^b |\widetilde{\gamma}'(t)|\, dt \geq \left|\int_a^b \widetilde{\gamma}'(t) dt\right| = |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|\,.$$

By rotating $\mathbb{R}^n$, we can assume that $x, y \in X$, where $X = \mathbb{R} \times \{0\}^{n-1}$ is the x^1-axis. Let $\widetilde{\gamma} : [a, b] \to \mathbb{R}^n$ be a piecewise smooth curve segment from x to y such that

$L_{\overline{g}}(\widetilde{\gamma}) = |y - x|$. Suppose that $\widetilde{\gamma}(c) \notin X$ for some $c \in (a, b)$. Then

$$\begin{aligned} L_{\overline{g}}(\widetilde{\gamma}) &= \int_a^c |\widetilde{\gamma}'(t)|\ dt + \int_c^b |\widetilde{\gamma}'(t)|\ dt \\ &\geq |\widetilde{\gamma}(c) - \widetilde{\gamma}(a)| + |\widetilde{\gamma}(b) - \widetilde{\gamma}(c)| \\ &> |y - x| = L_{\overline{g}}(\gamma), \end{aligned}$$

which is a contradiction. Therefore $\widetilde{\gamma}([a, b]) \subseteq X$. Let $\pi : \mathbb{R}^n \to X$ be the canonical projection. A similar argument shows that $\pi \circ \widetilde{\gamma}$ is a diffeomorphism from $[a, b]$ to the line segment from $\pi(x)$ to $\pi(y)$. □

Example 227. [Problem 13-11] Let $M = \mathbb{R}^2 \setminus \{0\}$, and let g be the restriction to M of the Euclidean metric $\overline{g}$. Show that there are points $p, q \in M$ for which there is no piecewise smooth curve segment γ from p to q in M with $L_g(\gamma) = d_g(p, q)$.

Let $p = (-1, 0)$ and $q = (1, 0)$. For each $0 < r < 1$, let

$$\begin{aligned} p' &= (-r, 0), \quad q' = (r, 0), \\ p'' &= (-r, r), \quad q'' = (r, r) \end{aligned}$$

and define $\alpha_r : [0, 5] \to M$ by

$$\alpha_r(t) = \begin{cases} (1-t)p + tp', & 0 \leq t < 1, \\ (2-t)p' + (t-1)p'', & 1 \leq t < 2, \\ (3-t)p'' + (t-2)q'', & 2 \leq t < 3, \\ (4-t)q'' + (t-3)q', & 3 \leq t < 4, \\ (5-t)q' + (t-4)q, & 4 \leq t \leq 5. \end{cases}$$

We have

$$\begin{aligned} L_g(\alpha_r) &= 1 - r + 4r + 1 - r = 2 + 2r \\ &\to 2 \end{aligned}$$

as $r \to 0$. Therefore $d_g(p, q) \leq 2$. But Theorem 226 shows that $d_g(p, q) \geq 2$, so $d_g(p, q) = 2$. There is no piecewise smooth curve segment γ from p to q such that $L_g(\gamma) = 2$, due to Theorem 226.

Theorem 228. *[Problem 13-12] Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric $\overline{g}$.*

(1) *Suppose $U, V \subseteq \mathbb{R}^n$ are connected open sets, $\varphi, \psi : U \to V$ are Riemannian isometries, and for some $p \in U$ they satisfy $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi = \psi$.*
(2) *The set of maps from $\mathbb{R}^n$ to itself given by the action of $\mathrm{E}(n)$ on $\mathbb{R}^n$ described in Example 7.32 is the full group of Riemannian isometries of $(\mathbb{R}^n, \overline{g})$.*

Proof. By considering $\varphi \circ \psi^{-1} : V \to V$, it suffices to show that for any connected open set $U \subseteq \mathbb{R}^n$ and any Riemannian isometry $f : U \to U$ satisfying $f(p) = p$ and $df_p = \mathrm{Id}_{T_pU}$ for some $p \in U$, we have $f = \mathrm{Id}_U$. If $\gamma : I \to U$ is a smooth curve then $L_{\overline{g}}(\gamma) = L_{\overline{g}}(f \circ \gamma)$, so Theorem 226 implies that f takes straight line segments to straight line segments (of the same length). Let

$$E = \{x \in U : f(x) = x,\ df_x = \mathrm{Id}_{T_xU}\};$$

then E is nonempty. Let $x \in E$ and let $B = B_r(x)$ be an open ball of radius r around x such that $\overline{B} \subseteq U$. For any $v \in \mathbb{S}^{n-1}$, let $\gamma : [-1,1] \to U$ be the curve $t \mapsto x + trv$ and let $L = \gamma([-1,1])$. Then $f \circ \gamma$ is a straight line segment of length $2r$ such that $(f \circ \gamma)(0) = x$. But $d(f \circ \gamma)_0 = df_x \circ d\gamma_0 = d\gamma_0$, so $(f \circ \gamma)([-1,1]) = L$. Using the fact that

$$d_{\overline{g}}(x, y) = d_{\overline{g}}(f(x), f(y)) = d_{\overline{g}}(x, f(y))$$

for all $y \in U$, we conclude that $f \circ \gamma = \gamma$. But v was arbitrary, so $f|_{\overline{B}}$ is the identity map on $\overline{B}(x)$. This shows that $B \subseteq E$, and therefore E is open. It is clear from continuity that E is closed. Since U is connected, we must have $E = U$. This proves (1).

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a Riemannian isometry. Then $\widetilde{\varphi}(x) = D\varphi(0)^{-1}(\varphi(x) - \varphi(0))$ defines a Riemannian isometry satisfying $D\widetilde{\varphi}(0) = I$ and $\widetilde{\varphi}(0) = 0$, so (1) shows that $\widetilde{\varphi} = \mathrm{Id}_{\mathbb{R}^n}$. Therefore $\varphi(x) = \varphi(0) + D\varphi(0)x$ is an affine transformation. □

Theorem 229. *[Problem 13-15] Let (M, g) be a Riemannian manifold, and let $\widehat{g}$ be the product metric on $M \times \mathbb{R}$ determined by g and the Euclidean metric on $\mathbb{R}$. Let $X = 0 \oplus d/dt$ be the product vector field on $M \times \mathbb{R}$ determined by the zero vector field on M and the standard coordinate vector field d/dt on $\mathbb{R}$. Then X is a Killing vector field for $(M \times \mathbb{R}, \widehat{g})$.*

Proof. The flow of X is given by $\theta((x, r), t) = (x, r + t)$. □

Theorem 230. *[Problem 13-16] If $g = f(t)dt^2$ is a Riemannian metric on $\mathbb{R}$, then g is complete if and only if both of the following improper integrals diverge:*

$$(*) \qquad \int_0^\infty \sqrt{f(t)}\, dt, \quad \int_{-\infty}^0 \sqrt{f(t)}\, dt.$$

Proof. If $x, y \in \mathbb{R}$ and $\gamma : [x, y] \to \mathbb{R}$ is the identity map then

$$d_g(x, y) = \int_x^y |\gamma'(t)|_g\, dt = \int_x^y |1|_g\, dt = \int_x^y \sqrt{f(t)}\, dt.$$

Note that $f(t) > 0$ for all $t \in \mathbb{R}$. If $\int_0^\infty \sqrt{f(t)}\, dt$ converges, then $\{x_n\} = \{1, 2, \ldots\}$ is a Cauchy sequence. Suppose that g is complete; then $n \to x$ for some $x \in \mathbb{R}$. If $x \geq 0$ then $d_g(n, x) = \int_x^n \sqrt{f(t)}\, dt$ converges to a positive number as $n \to \infty$, which is a contradiction. A similar contradiction arises if $x < 0$. Therefore g is not complete. Conversely, suppose that both integrals in (*) diverge and let $\{x_n\}$ be a Cauchy sequence. Then $\{x_n\}$ is contained in a set $K \subseteq \mathbb{R}$ that is bounded with respect to the metric induced by g. Since the integrals in (*) diverge, K must also be bounded with respect to the Euclidean metric. By replacing K with a closed and bounded interval containing K, we may assume that K is compact. There are positive constants c, C such that

$$c \leq \sqrt{f(t)} \leq C \quad \text{and} \quad \sqrt{f(t)}/C \leq 1 \leq \sqrt{f(t)}/c$$

for all $t \in K$. Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists an integer N such that

$$-c\varepsilon < \int_{x_m}^{x_n} \sqrt{f(t)}\, dt < c\varepsilon$$

for all $m, n \geq N$. Then

$$-\varepsilon < -\frac{c\varepsilon}{C} < \frac{1}{C}\int_{x_m}^{x_n} \sqrt{f(t)}\, dt \leq |x_n - x_m| \leq \frac{1}{c}\int_{x_m}^{x_n} \sqrt{f(t)}\, dt < \frac{c\varepsilon}{c} = \varepsilon,$$

so in the Euclidean metric, $\{x_n\}$ is Cauchy and $x_n \to x$ for some $x \in K$. We have

$$\left| \int_{x_n}^{x} \sqrt{f(t)}\, dt \right| \to 0$$

as $n \to \infty$, so $x_n \to x$ in the metric induced by g. □

Theorem 231. *[Problem 13-18] Suppose (M, g) is a connected Riemannian manifold, $S \subseteq M$ is a connected embedded submanifold, and $\widetilde{g}$ is the induced Riemannian metric on S.*

(1) *$d_{\widetilde{g}}(p, q) \geq d_g(p, q)$ for $p, q \in S$.*
(2) *If (M, g) is complete and S is properly embedded, then $(S, \widetilde{g})$ is complete.*
(3) *Every connected smooth manifold admits a complete Riemannian metric.*

Proof. Let $\iota : S \hookrightarrow M$ be the inclusion map. If $\gamma : [a, b] \to S$ is a piecewise smooth curve segment from p to q in S then $\iota \circ \gamma$ is a piecewise smooth curve segment from p to q in M and $L_{\widetilde{g}}(\gamma) = L_g(\iota \circ \gamma) \geq d_g(p, q)$. Since this holds for all γ, we have $d_{\widetilde{g}}(p, q) \geq d_g(p, q)$. This proves (1). Let $\{x_n\}$ be a Cauchy sequence in S and let $\varepsilon > 0$. There exists an integer N such that

$$d_g(x_m, x_n) \leq d_{\widetilde{g}}(x_m, x_n) < \varepsilon$$

for all $m, n \geq N$, so $\{x_n\}$ is a Cauchy sequence in M and $x_n \to x$ for some $x \in M$. But S is closed in M, so $x \in S$. This proves (2). Let N be a connected smooth manifold. Theorem 6.15 shows that there is a proper smooth embedding $j : N \hookrightarrow \mathbb{R}^{2n+1}$. Since $(\mathbb{R}^{2n+1}, \overline{g})$ is complete, $(N, j^*\overline{g})$ is complete by (2). This proves (3). □

Example 232. [Problem 13-19] The following example shows that the converse of Theorem 231(b) does not hold. Define $F : \mathbb{R} \to \mathbb{R}^2$ by $F(t) = ((e^t+1)\cos t, (e^t+1)\sin t)$. Show that F is an embedding that is not proper, yet $\mathbb{R}$ is complete in the metric induced from the Euclidean metric on $\mathbb{R}^2$.

We have
$$DF(t) = \begin{bmatrix} e^t(\cos t - \sin t) - \sin t \\ e^t(\cos t + \sin t) + \cos t \end{bmatrix},$$
which is never zero. Since F is an open map, it is a smooth embedding. But $F^{-1}\left(\overline{B}_2(0)\right)$ is not compact, where $\overline{B}_2(0)$ is the closed ball of radius 2 around $(0,0)$. The metric on $\mathbb{R}$ induced from the Euclidean metric on $\mathbb{R}^2$ is given by
$$\begin{aligned} f^*\overline{g} &= d((e^t+1)\cos t)^2 + d((e^t+1)\sin t)^2 \\ &= (2e^{2t} + 2e^t + 1)dt^2. \end{aligned}$$
But
$$\int_0^\infty \sqrt{2e^{2t} + 2e^t + 1}\, dt \quad \text{and} \quad \int_{-\infty}^0 \sqrt{2e^{2t} + 2e^t + 1}\, dt$$
diverge, so Theorem 230 shows that $f^*\overline{g}$ is complete.

Theorem 233. *[Problem 13-21] Let (M, g) be a Riemannian manifold, let $f \in C^\infty(M)$, and let $p \in M$ be a regular point of f.*

(1) *Among all unit vectors $v \in T_pM$, the directional derivative vf is greatest when v points in the same direction as $\operatorname{grad} f|_p$, and the length of $\operatorname{grad} f|_p$ is equal to the value of the directional derivative in that direction.*
(2) $\operatorname{grad} f|_p$ *is normal to the level set of f through p.*

Proof. We have $\langle \operatorname{grad} f|_p, v\rangle = vf$, so the Cauchy-Schwarz inequality shows that
$$|\left\langle \operatorname{grad} f|_p, v\right\rangle_g|$$
is maximized when $v = \pm \operatorname{grad} f|_p$. If S is the level set of f through p then $T_pS = \ker df_p$ by Proposition 5.38. Clearly $\langle \operatorname{grad} f|_p, v\rangle = vf = df_p(v) = 0$ for every $v \in \ker df_p$. □

Theorem 234. *[Problem 13-22] For any smooth manifold M with or without boundary, the vector bundles TM and T^*M are smoothly isomorphic over M.*

Proof. Choose any Riemannian metric g on M and let $\widehat{g} : TM \to T^*M$ be the smooth bundle homomorphism defined at each $p \in M$ by $\widehat{g}(v)(w) = g_p(v, w)$. Then $\widehat{g}$ is injective (and therefore bijective) at each point, so Theorem 174 shows that $\widehat{g}$ is a smooth bundle isomorphism. □

Example 235. [Problem 13-23] Is there a smooth covector field on $\mathbb{S}^2$ that vanishes at exactly one point?

Yes, due to Theorem 128 and Theorem 234.

Chapter 14. Differential Forms

Theorem 236. *[Exercise 14.4] Let α be a covariant tensor on a finite-dimensional vector space.*

(1) $\operatorname{Alt}\alpha$ *is alternating.*
(2) $\operatorname{Alt}\alpha = \alpha$ *if and only if* α *is alternating.*

Proof. If $\tau \in S_k$ then

$$\begin{aligned}
(\operatorname{Alt}\alpha)(v_{\tau(1)}, \dots, v_{\tau(k)}) &= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\alpha(v_{(\sigma\circ\tau)(1)}, \dots, v_{(\sigma\circ\tau)(k)}) \\
&= \operatorname{sgn}(\tau)\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma\circ\tau)\alpha(v_{(\sigma\circ\tau)(1)}, \dots, v_{(\sigma\circ\tau)(k)}) \\
&= \operatorname{sgn}(\tau)\frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
&= \operatorname{sgn}(\tau)(\operatorname{Alt}\alpha)(v_1, \dots, v_k),
\end{aligned}$$

which proves (1). If α is alternating then

$$\begin{aligned}
(\operatorname{Alt}\alpha)(v_1, \dots, v_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
&= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)^2\alpha(v_1, \dots, v_k) \\
&= \alpha(v_1, \dots, v_k).
\end{aligned}$$

Conversely, if $\operatorname{Alt}\alpha = \alpha$ then α is alternating since $\operatorname{Alt}\alpha$ is alternating. This proves (2). ∎

Theorem 237. *[Exercise 14.12] The wedge product is the unique associative, bilinear, and anticommutative map $\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$ satisfying*

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I$$

for any basis (ε^i) for V^* and any multi-index $I = (i_1, \dots, i_k)$.

Proof. Let $\widetilde{\wedge}$ be such a map. Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$. By Lemma 14.10,

$$\begin{aligned}
\omega \widetilde{\wedge} \eta &= \left({\sum_I}' \omega_I \varepsilon^I \right) \widetilde{\wedge} \left({\sum_J}' \eta_J \varepsilon^J \right) \\
&= {\sum_I}' {\sum_J}' \omega_I \eta_J \varepsilon^I \widetilde{\wedge} \varepsilon^J \\
&= {\sum_I}' {\sum_J}' \omega_I \eta_J \varepsilon^{i_1} \widetilde{\wedge} \cdots \widetilde{\wedge} e^{i_k} \widetilde{\wedge} e^{j_1} \widetilde{\wedge} \cdots \widetilde{\wedge} e^{j_l} \\
&= {\sum_I}' {\sum_J}' \omega_I \eta_J \varepsilon^{IJ} \\
&= {\sum_I}' {\sum_J}' \omega_I \eta_J \varepsilon^I \wedge \varepsilon^J \\
&= \left({\sum_I}' \omega_I \varepsilon^I \right) \wedge \left({\sum_J}' \eta_J \varepsilon^J \right) \\
&= \omega \wedge \eta.
\end{aligned}$$

□

Theorem 238. *[Exercise 14.14] $\Lambda^k T^* M$ is a smooth subbundle of $T^k T^* M$, and therefore is a smooth vector bundle of rank $\binom{n}{k}$ over M.*

Proof. This follows from Proposition 14.8 and Lemma 10.32. □

Theorem 239. *[Exercise 14.28] The diagram below commutes:*

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3).
\end{tikzcd}$$

Therefore $\mathrm{curl} \circ \mathrm{grad} = 0$ *and* $\mathrm{div} \circ \mathrm{curl} = 0$ *on* $\mathbb{R}^3$*. The analogues of the left-hand and right-hand square commute when $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ for any n.*

Proof. In $\mathbb{R}^n$, we have

$$(\flat \circ \operatorname{grad})(f) = \flat\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}\right) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i = df$$

and

$$\begin{aligned}
(d\circ\beta)(X) &= d(X \lrcorner (dx^1\wedge\cdots\wedge dx^n)) \\
&= d\left(\sum_{i=1}^n (-1)^{i-1} dx^i(X) dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\right) \\
&= d\left(\sum_{i=1}^n (-1)^{i-1} X^i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\right) \\
&= \sum_{i=1}^n (-1)^{i-1}\sum_{j=1}^n \frac{\partial X^i}{\partial x^j} dx^j\wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n \\
&= \sum_{i=1}^n \frac{\partial X^i}{\partial x^i} dx^1\wedge\cdots\wedge dx^n \\
&= *\left(\sum_{i=1}^n \frac{\partial X^i}{\partial x^i}\right) \\
&= (*\circ\operatorname{div})(X).
\end{aligned}$$

In $\mathbb{R}^3$ we have

$$\begin{aligned}
(\beta\circ\operatorname{curl})(X) &= \beta\left(\left(\frac{\partial X^3}{\partial x^2}-\frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1}+\left(\frac{\partial X^1}{\partial x^3}-\frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2}\right. \\
&\qquad \left. +\left(\frac{\partial X^2}{\partial x^1}-\frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}\right) \\
&= \left(\frac{\partial X^2}{\partial x^1}-\frac{\partial X^1}{\partial x^2}\right)dx^1\wedge dx^2+\left(\frac{\partial X^3}{\partial x^1}-\frac{\partial X^1}{\partial x^3}\right)dx^1\wedge dx^3 \\
&\qquad +\left(\frac{\partial X^3}{\partial x^2}-\frac{\partial X^2}{\partial x^3}\right)dx^2\wedge dx^3 \\
&= d\left(X^1dx^1+X^2dx^2+X^3dx^3\right) \\
&= (d\circ\flat)(X).
\end{aligned}$$

□

Theorem 240. *[Problem 14-1] Covectors $\omega^1,\dots,\omega^k\in V^*$ are linearly dependent if and only if $\omega^1\wedge\cdots\wedge\omega^k=0$.*

Proof. Let $n = \dim V$. Suppose that $\omega^1, \dots, \omega^k$ are linearly dependent and assume without loss of generality that $a_1\omega^1 + \cdots + a_k\omega^k = 0$ with $a_1 \neq 0$. Then $\omega^1 = -a_1^{-1}(a_2\omega^2 + \cdots + a_k\omega^k)$, so

$$\omega^1 \wedge \cdots \wedge \omega^k = -a_1^{-1}(a_2\omega^2 + \cdots + a_k\omega^k) \wedge \omega^2 \wedge \cdots \wedge \omega^k = 0.$$

Conversely, suppose that $\omega^1, \dots, \omega^k$ are linearly independent. We can extend $\{\omega^1, \dots, \omega^k\}$ to a basis $\{\omega^1, \dots, \omega^n\}$ of $\Lambda^n(V^*)$. Then

$$\omega^1 \wedge \cdots \wedge \omega^n = (\omega^1 \wedge \cdots \wedge \omega^k) \wedge (\omega^{k+1} \wedge \cdots \wedge \omega^n) \neq 0,$$

so $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$. □

Theorem 241. *[Problem 14-5] Let M be a smooth n-manifold with or without boundary, and let $(\omega^1, \dots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $(\omega^1|_p, \dots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \dots, \alpha^k$ on U such that*

$$\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0,$$

each α^i can be written as a linear combination of $\omega^1, \dots, \omega^k$ with smooth coefficients.

Proof. First note that each α^i is in the span of $\omega^1, \dots, \omega^k$, since

$$\begin{aligned}
\alpha^i \wedge \omega^1 \wedge \cdots \wedge \omega^k &= (-1)^{i+1}\omega^1 \wedge \cdots \wedge \alpha^i \wedge \omega^i \wedge \cdots \wedge \omega^k \\
&= (-1)^i \omega^1 \wedge \cdots \wedge \sum_{j \neq i} \alpha^j \wedge \omega^j \wedge \cdots \wedge \omega^k \\
&= 0.
\end{aligned}$$

Therefore we have $\alpha^i = r_{ij}\omega^j$ for some coefficients $r_{ij} : U \to \mathbb{R}$. But the span of $\omega^1, \dots, \omega^k$ forms a smooth subbundle of TM, so Proposition 10.22 shows that the functions r_{ij} are smooth. □

Example 242. [Problem 14-6] Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

(1) Compute ω in spherical coordinates (ρ, φ, θ) defined by
$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$
(2) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
(3) Compute the pullback $\iota_{\mathbb{S}^2}^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.
(4) Show that $\iota_{\mathbb{S}^2}^*\omega$ is nowhere zero.

We have

$$\begin{aligned} dx &= \sin\varphi\cos\theta\,d\rho + \rho\cos\varphi\cos\theta\,d\varphi - \rho\sin\varphi\sin\theta\,d\theta, \\ dy &= \sin\varphi\sin\theta\,d\rho + \rho\cos\varphi\sin\theta\,d\varphi + \rho\sin\varphi\cos\theta\,d\theta, \\ dz &= \cos\varphi\,d\rho - \rho\sin\varphi\,d\varphi, \end{aligned}$$

so

$$\omega = \rho^3 \sin\varphi\,d\varphi \wedge d\theta.$$

Also,

$$d\omega = 3dx \wedge dy \wedge dz$$

and

$$d\omega = 3\rho^2 \sin\varphi\,d\rho \wedge d\varphi \wedge d\theta.$$

The pullback of ω to $\mathbb{S}^2$ is given by

$$\iota_{\mathbb{S}^2}^*\omega = \sin\varphi\,d\varphi \wedge d\theta,$$

which is defined for $(\varphi, \theta) \in (0, \pi) \times (0, 2\pi)$. This expression is never zero since $\sin\varphi > 0$ for $\varphi \in (0, \pi)$.

Chapter 15. Orientations

Theorem 243. *[Exercise 15.4] Suppose M is an oriented smooth n-manifold with or without boundary, and $n \geq 1$. Every local frame with connected domain is either positively oriented or negatively oriented.*

Proof. Let (E_i) be a local frame with a connected domain U. Let D be the points of U at which (E_i) is positively oriented. Let $p \in D$ and let $(\widetilde{E}_i)$ be an oriented local frame defined on a neighborhood $V \subseteq U$ of p. Writing $E_i = B_i^j \widetilde{E}_j$ for continuous component functions $B_i^j : V \to \mathbb{R}$, we have $\det(B_i^j(p)) > 0$. Since det is continuous, there is a neighborhood of p on which E_i is oriented. This shows that D is open. A similar argument shows that $U \setminus D$ is open, so D is either empty or equal to U by the connectedness of U. □

Theorem 244. *[Exercise 15.8] Suppose $M_1, \ldots, M_k$ are orientable smooth manifolds. There is a unique orientation on $M_1 \times \cdots \times M_k$, called the **product orientation**, with the following property: if for each $i = 1, \ldots, k$, ω_i is an orientation form for the given orientation on M_i, then $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k$ is an orientation form for the product orientation.*

Proof. Let $\eta_1, \dots, \eta_k$ be orientation forms for $M_1, \dots, M_k$ and let $\mathcal{O}$ be the orientation determined by $\pi_1^*\eta_1 \wedge \cdots \wedge \pi_k^*\eta_k$. Suppose $\omega_1, \dots, \omega_k$ are orientation forms for $M_1, \dots, M_k$. Then $\omega_i = f_i\eta_i$ for some strictly positive continuous functions $f_i : M_i \to \mathbb{R}$, so

$$\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k = (f_1 \circ \pi_1) \cdots (f_k \circ \pi_k)\eta_1 \wedge \cdots \wedge \eta_k.$$

Since $(f_1 \circ \pi_1) \cdots (f_k \circ \pi_k)$ is strictly positive, $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k$ determines the same orientation as $\eta_1 \wedge \cdots \wedge \eta_k$. □

Theorem 245. *[Exercise 15.10] Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, they are equal.*

Proof. If $\dim M = 0$ then M consists of a single point, so the result is clear. Suppose $\mathcal{O}, \mathcal{O}'$ are two orientations of M that agree at a single point p. Let $D = \{x \in M : \mathcal{O}_x = \mathcal{O}'_x\}$; this set is nonempty since $p \in D$. Let $x \in D$ and let (E_i) be a local frame defined in a connected neighborhood of x that is positively oriented with respect to $\mathcal{O}'$. Applying Theorem 243 to $(M, \mathcal{O})$ shows that (E_i) is positively oriented with respect to $\mathcal{O}$. Therefore D is open. A similar argument shows that $M \setminus D$ is open, so $D = M$ by the connectedness of M.

Since M is orientable, there is an orientation $\mathcal{O}$ for M. Let ω be an orientation form for $\mathcal{O}$; then $-\omega$ determines an orientation $\mathcal{O}'$ for M with $\mathcal{O}_p \neq \mathcal{O}'_p$ for all $p \in M$. If $\mathcal{O}''$ is any orientation and p is any point in M then either $\mathcal{O}''_p = \mathcal{O}_p$ or $\mathcal{O}''_p = \mathcal{O}'_p$, so $\mathcal{O}'' = \mathcal{O}$ or $\mathcal{O}'' = \mathcal{O}'$. □

Theorem 246. *[Exercise 15.12] Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. The orientation of M restricts to an orientation of D. If ω is an orientation form for M, then $\iota_D^*\omega$ is an orientation form for D.*

Proof. It suffices to check that $\iota_D^*\omega$ is nonvanishing. But this is clear since $d(\iota_D)_p$ is an isomorphism for every $p \in M$. □

Theorem 247. *[Exercise 15.13] Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. The following are equivalent:*

1. *F is orientation-preserving.*
2. *With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.*
3. *For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.*

Proof. If $(U,(x^i))$ and $(V,(y^i))$ are oriented smooth charts for M and N respectively, then dF_p takes the oriented basis $(\partial/\partial x^i|_p)$ to an oriented basis

$$\left(\frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\bigg|_p\right).$$

Since $(\partial/\partial y^i|_p)$ is also an oriented basis, we have $\det(\partial F^j/\partial x^i(p)) > 0$. This proves (1) $\Rightarrow$ (2). Suppose that (2) holds. By shrinking U and V, we can assume that they are connected sets and that $F|_U : U \to V$ is a diffeomorphism. Then

$$\omega = f\,dy^1 \wedge \cdots \wedge dy^n$$

on V for some positive continuous function f, so by Proposition 14.20 we have

$$F^*\omega = (f \circ F)(\det DF)dx^1 \wedge \cdots \wedge dx^n.$$

Since $\det DF > 0$ by (2), $F^*\omega$ is positively oriented on U. This proves (2) $\Rightarrow$ (3). Finally, suppose that (3) holds and let (E_i) be an oriented basis of T_pM. If ω is a positively oriented orientation form for N then

$$(F^*\omega)_p(E_1,\ldots,E_n) > 0 \Rightarrow \omega_{F(p)}(dF_p(E_1),\ldots,dF_p(E_n)) > 0,$$

so $(dF_p(E_i))$ is an oriented basis of $T_{F(p)}N$. This proves (3) $\Rightarrow$ (1). □

Theorem 248. *[Exercise 15.14] A composition of orientation-preserving maps is orientation-preserving.*

Proof. Let $F : M \to N$ and $G : N \to P$ be orientation-preserving maps. If (E_i) is an oriented basis of T_pM then $(dF_p(E_i))$ is an oriented basis of $T_{F(p)}N$ and $(d(G\circ F)_p(E_i))$ is an oriented basis of $T_{(G\circ F)(p)}P$. □

Theorem 249. *[Exercise 15.16] Suppose $F : M \to N$ and $G : N \to P$ are local diffeomorphisms and $\mathcal{O}$ is an orientation on P. Then $(G \circ F)^*\mathcal{O} = F^*(G^*\mathcal{O})$.*

Proof. This is clear from the fact that $(G \circ F)^*\omega = F^*G^*\omega$ for any orientation form ω. □

Theorem 250. *[Exercise 15.20] Every Lie group has precisely two left-invariant orientations, corresponding to the two orientations of its Lie algebra.*

Proof. Let G be a Lie group. Corollary 8.39 shows that there are at least two left-invariant orientations. If $\mathcal{O}$ is a left-invariant orientation on G then it is completely determined by $\mathcal{O}_e$, so there are exactly two left-invariant orientations. □

Theorem 251. *[Exercise 15.30] Suppose (M,g) and $(\widetilde{M},\widetilde{g})$ are positive-dimensional Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Then $F^*\omega_{\widetilde{g}} = \omega_g$.*

Proof. Let $p \in M$ and let U be a neighborhood of p for which $F|_U$ is an isometry. If (E_i) is a local oriented orthonormal frame for M then $(dF(E_i))$ is a local oriented orthonormal frame for $\widetilde{M}$ on a neighborhood of $F(p)$ since

$$\langle dF(E_i), dF(E_j)\rangle_{\widetilde{g}} = \langle E_i, E_j\rangle_g = \delta_{ij}.$$

Therefore

$$(F^*\omega_{\widetilde{g}})(E_1, \ldots, E_n) = \omega_{\widetilde{g}}(dF(E_1), \ldots, dF(E_n)) = 1,$$

which shows that $F^*\omega_{\widetilde{g}}$ is the Riemannian volume form on M. ☐

Theorem 252. *[Problem 15-1] Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Then M is orientable.*

Proof. Denote the two open submanifolds by N and P. We can assume that there is some $p \in N \cap P$, for otherwise the result is trivial. Choose an orientation $\mathcal{O}$ for N, and choose an orientation $\mathcal{O}'$ for P such that $\mathcal{O}_p = \mathcal{O}'_p$. By Theorem 245, $\mathcal{O}$ and $\mathcal{O}'$ agree on $N \cap P$, and it is easy to check that the orientation given by

$$\mathcal{O}''_x = \begin{cases} \mathcal{O}_x, & x \in N, \\ \mathcal{O}'_x, & x \in P \end{cases}$$

is well-defined and continuous. ☐

Theorem 253. *[Problem 15-2] Suppose M and N are oriented smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. If M is connected, then F is either orientation-preserving or orientation-reversing.*

Proof. If M and N are 0-manifolds, then M and N both consist of a single point and the result is trivial. Let D be the set of points x for which dF_x takes oriented bases of T_xM to oriented bases of $T_{F(x)}N$. Let ω be a positively oriented orientation form for N. Let $x \in D$ and let (E_i) be an oriented local frame defined on a neighborhood U of x. Then $(F^*\omega)_x(E_1|_x, \ldots, E_n|_x) > 0$, so by continuity there is a neighborhood $V \subseteq U$ of x on which $(F^*\omega)(E_1, \ldots, E_n) > 0$. Therefore $V \subseteq D$ by Theorem 247, and D is open. A similar argument shows that $M \setminus D$ is open, so $D = \emptyset$ or $D = M$ since M is connected. ☐

Theorem 254. *[Problem 15-3] Suppose $n \geq 1$, and let $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Then α is orientation-preserving if and only if n is odd.*

Proof. The outward unit normal vector field along $\mathbb{S}^n$ is $N = x^i\partial/\partial x^i$, and Corollary 15.34 shows that the Riemannian volume form $\omega_{\widetilde{g}}$ of $\mathbb{S}^n$ is

$$\omega_{\widetilde{g}} = \iota^*_{\mathbb{S}^n}(N \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1})) = \iota^*_{\mathbb{S}^n}\left(\sum_{i=1}^{n+1}(-1)^{i-1}x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}\right).$$

We have

$$\begin{aligned}\alpha^*\omega_{\widetilde{g}} &= \iota_{\mathbb{S}^n}^*\left(\sum_{i=1}^{n+1}(-1)^{i-1}(-x^i)\,(-dx^1)\wedge\cdots\wedge\widehat{(-dx^i)}\wedge\cdots\wedge(-dx^{n+1})\right)\\ &= (-1)^{n+1}\omega_{\widetilde{g}},\end{aligned}$$

so F is orientation-preserving if and only if n is odd. □

Theorem 255. *[Problem 15-5] Let M be a smooth manifold with or without boundary. The total spaces of TM and T^*M are orientable.*

Proof. Let $\pi : TM \to M$ be the projection. Let (U, φ) and (V, ψ) be smooth charts for M, and let $(\pi^{-1}(U), \widetilde{\varphi})$ and $(\pi^{-1}(V), \widetilde{\psi})$ be the corresponding charts for TM. The transition map is given by

$$(\widetilde{\psi}\circ\widetilde{\varphi}^{-1})(x^1,\dots,x^n,v^1,\dots,v^n) = \left(\widetilde{x}^1(x),\dots,\widetilde{x}^n(x),\frac{\partial\widetilde{x}^1}{\partial x^j}(x)v^j,\dots,\frac{\partial\widetilde{x}^n}{\partial x^j}(x)v^j\right).$$

The Jacobian matrix of this map at $(x^1,\dots,x^n,v^1,\dots,v^n)$ is

$$\begin{bmatrix}\frac{\partial\widetilde{x}^i}{\partial x^j}(x) & 0\\ * & \frac{\partial\widetilde{x}^i}{\partial x^j}(x)\end{bmatrix},$$

which clearly has positive determinant. The result follows from Proposition 15.6. A similar computation holds for T^*M. □

Theorem 256. *[Problem 15-7] Suppose M is an oriented Riemannian manifold with or without boundary, and $S \subseteq M$ is an oriented smooth hypersurface with or without boundary. There is a unique smooth unit normal vector field along S that determines the given orientation of S.*

Proof. Let $n = \dim M$, let ω be an orientation form for M, and let $\mathcal{O}$ be the orientation on S. Let NS be the normal bundle of S (see Theorem 213), which is of rank 1. Let Z be the set of all zero vectors in NS. Define a map $f : NS \setminus Z \to \{-1, +1\}$ as follows: if $v \in N_pS$ then choose an oriented (with respect to $\mathcal{O}$) basis (E_i) for T_pS and define

$$f(v) = \frac{\omega(v, E_1, \dots, E_{n-1})}{|\omega(v, E_1, \dots, E_{n-1})|}.$$

This map is well-defined because (E_i) is oriented, and is continuous because there is an oriented local frame defined on a neighborhood of every p. Let $P = f^{-1}(\{+1\})$. It is easy to see that P is convex and open in NS, so Theorem 217 shows that there is a smooth global section $X : S \to P$. Then $\widetilde{X} = X/|X|_g$ is the desired smooth unit normal vector field along S. □

Theorem 257. *[Problem 15-8] Suppose M is an orientable Riemannian manifold, and $S \subseteq M$ is an immersed or embedded submanifold with or without boundary.*

(1) *If S has trivial normal bundle, then S is orientable.*
(2) *If S is an orientable hypersurface, then S has trivial normal bundle.*

Proof. If S has trivial normal bundle then there is a global frame for NS, and Proposition 15.21 shows that S is orientable. If S is an orientable hypersurface, then Theorem 256 gives a global frame for NS (since NS is of rank 1). □

Theorem 258. *[Problem 15-9] Let S be an oriented, embedded, 2-dimensional submanifold with boundary in $\mathbb{R}^3$, and let $C = \partial S$ with the induced orientation. By Theorem 256, there is a unique smooth unit normal vector field N on S that determines the orientation. Let T be the oriented unit tangent vector field on C, and let V be the unique unit vector field tangent to S along C that is orthogonal to T and inward-pointing. Then (T_p, V_p, N_p) is an oriented orthonormal basis for $\mathbb{R}^3$ at each $p \in C$.*

Proof. It is clear that (T_p, V_p, N_p) is an orthonormal basis, so it remains to check that (T_p, V_p, N_p) is oriented. This is true if and only if (N_p, T_p, V_p) is oriented, and by the definition of the induced orientation, this is true if and only if (T_p, V_p) is an oriented basis for T_pS. This is true since $(-V_p, T_p)$ is an oriented basis for T_pS. □

Theorem 259. *[Problem 15-10] Let M be a connected nonorientable smooth manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering. If X is any oriented smooth manifold with or without boundary, and $F : X \to M$ is any local diffeomorphism, then there exists a unique orientation-preserving local diffeomorphism $\widehat{F} : X \to \widehat{M}$ such that $\widehat{\pi} \circ \widehat{F} = F$.*

Proof. Define $\widehat{F}(x) = (F(x), \mathcal{O}_{F(x)})$, where $\mathcal{O}_{F(x)}$ is the orientation of $T_{F(x)}M$ determined by F and the orientation of X. It is clear that $\widehat{\pi} \circ \widehat{F} = F$. Let $\widehat{U}_{\mathcal{O}}$ be an element of the basis defined in Proposition 15.40. If U is connected, then $\widehat{F}^{-1}(\widehat{U}_{\mathcal{O}})$ is either U or empty. Therefore $\widehat{F}^{-1}(\widehat{U}_{\mathcal{O}})$ is open for any open set U, and $\widehat{F}$ is continuous. By Theorem 41, $\widehat{F}$ is smooth, and it is easy to check that it is orientation-preserving and a local diffeomorphism. Uniqueness follows from the fact that $\widehat{F}$ is orientation-preserving. □

Theorem 260. *[Problem 15-11] Let M be a nonorientable connected smooth manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering. If $\widetilde{M}$ is an oriented smooth manifold with or without boundary that admits a two-sheeted smooth covering map $\widetilde{\pi} : \widetilde{M} \to M$, then there exists a unique orientation-preserving diffeomorphism $\varphi : \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \widetilde{\pi}$.*

Proof. By Theorem 259, there is a unique orientation-preserving local diffeomorphism $\varphi : \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \widetilde{\pi}$. Similarly, there is a unique orientation-preserving local diffeomorphism $\psi : \widehat{M} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \psi = \widehat{\pi}$. Then $\widehat{\pi} \circ \varphi \circ \psi = \widetilde{\pi} \circ \psi = \widehat{\pi}$, so $\varphi \circ \psi = \mathrm{Id}_{\widehat{M}}$ by uniqueness. Similarly, $\psi \circ \varphi = \mathrm{Id}_{\widetilde{M}}$. Therefore φ is a diffeomorphism. $\square$

Theorem 261. *[Problem 15-22] Every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be such a diffeomorphism; then $f'(x) < 0$ for all $x \in \mathbb{R}$. Let $g(x) = f(x) - x$; then $g'(x) = f'(x) - 1 < -1$ for all $x \in \mathbb{R}$. If $g(0) = 0$, then we are done. If $g(0) > 0$ then by the mean value theorem, there exists a point $t \in (0, g(0))$ such that

$$\frac{g(g(0)) - g(0)}{g(0)} = g'(t) < -1,$$

i.e. $g(g(0)) < 0$, and the intermediate value theorem shows that $g(x) = 0$ for some $x \in (0, g(0))$. Similarly, if $g(0) < 0$ then there exists a point $t \in (g(0), 0)$ such that

$$\frac{g(g(0)) - g(0)}{g(0)} = g'(t) < -1,$$

i.e. $g(g(0)) > 0$, and the intermediate value theorem shows that $g(x) = 0$ for some $x \in (g(0), 0)$. $\square$

Chapter 16. Integration on Manifolds

Theorem 262. *[Exercise 16.29] Suppose (M, g) is an oriented Riemannian manifold and $f : M \to \mathbb{R}$ is continuous and compactly supported. Then*

$$\left| \int_M f \, dV_g \right| \leq \int_M |f| \, dV_g.$$

Proof. If f is supported in the domain of a single oriented smooth chart (U, φ), then

$$\begin{aligned} \left| \int_M f \, dV_g \right| &= \left| \int_{\varphi(U)} f(x) \sqrt{\det(g_{ij})} \, dx^1 \cdots dx^n \right| \\ &\leq \int_{\varphi(U)} |f(x)| \sqrt{\det(g_{ij})} \, dx^1 \cdots dx^n \\ &= \int_M |f| \, dV_g. \end{aligned}$$

For the general case, cover $\operatorname{supp} f$ by finitely many domains of positively or negatively oriented smooth charts $\{U_i\}$ and choose a subordinate smooth partition of unity $\{\psi_i\}$. Then

$$\begin{aligned}\left|\int_M f\,dV_g\right| &= \left|\sum_i \int_M \psi_i f\,dV_g\right| \\ &\leq \sum_i \int_M \psi_i\,|f|\;dV_g \\ &= \int_M |f|\;dV_g.\end{aligned}$$

□

Example 263. [Problem 16-2] Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w, x, y, z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = xyz\,dw \wedge dy.$$

Let $D = (0, 2\pi) \times (0, 2\pi)$ and define an orientation-preserving diffeomorphism $F : D \to \mathbb{T}^2$ by

$$F(\theta, \varphi) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$$

so that

$$\begin{aligned} dw &= -\sin\theta\,d\theta, \quad dx = \cos\theta\,d\theta, \\ dy &= -\sin\varphi\,d\varphi, \quad dz = \cos\varphi\,d\varphi. \end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{T}^2} \omega &= \int_{\mathbb{T}^2} \sin\theta\cos\varphi\sin\varphi(-\sin\theta\,d\theta) \wedge (-\sin\varphi\,d\varphi) \\ &= \int_{\mathbb{T}^2} \sin^2\theta\cos\varphi\sin^2\varphi\,d\theta \wedge d\varphi \\ &= \left(\int_0^{2\pi} \sin^2\theta\,d\theta\right)\left(\int_0^{2\pi} \cos\varphi\sin^2\varphi\,d\varphi\right) \\ &= \pi \cdot 0 \\ &= 0.\end{aligned}$$

Theorem 264. *[Problem 16-3] Suppose E and M are smooth n-manifolds with or without boundary, and $\pi : E \to M$ is a smooth k-sheeted covering map or generalized covering map.*

(1) *If E and M are oriented and π is orientation-preserving, then $\int_E \pi^*\omega = k\int_M \omega$ for any compactly supported n-form ω on M.*
(2) *$\int_E \pi^*\mu = k\int_M \mu$ whenever μ is a compactly supported density on M.*

Proof. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}\omega$ by evenly covered sets, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. For each i, let $U_i^{(1)}, \ldots, U_i^{(k)}$ be the components of $\pi^{-1}(U_i)$. Let $\left\{\widehat{\psi}_i^{(j)}\right\}$ be a smooth partition of unity subordinate to the open cover $\left\{U_i^{(j)}\right\}$ of $\operatorname{supp}\pi^*\omega$. Applying Proposition 16.6(d), we have

$$\begin{aligned}
k\int_M \omega &= \sum_i \sum_{j=1}^k \int_{U_i} \psi_i\omega \\
&= \sum_i \sum_{j=1}^k \int_{U_i^{(j)}} (\psi_i \circ \pi|_{U_i^{(j)}})\pi^*\omega \\
&= \sum_i \sum_{j=1}^k \int_{U_i^{(j)}} (\psi_i \circ \pi|_{U_i^{(j)}})\pi^*\omega \\
&= \sum_i \sum_{j=1}^k \int_{U_i^{(j)}} \sum_{i'} \sum_{j'=1}^k \widehat{\psi}_{i'}^{(j')} (\psi_i \circ \pi|_{U_i^{(j)}})\pi^*\omega \\
&= \int_E \sum_{i'} \sum_{j'=1}^k \widehat{\psi}_{i'}^{(j')} \sum_i \sum_{j=1}^k (\psi_i \circ \pi|_{U_i^{(j)}})\pi^*\omega \\
&= \int_E \sum_{i'} \sum_{j'=1}^k \widehat{\psi}_{i'}^{(j')} \pi^*\omega \\
&= \int_E \pi^*\omega.
\end{aligned}$$

This proves (1). Part (2) is similar. $\square$

Theorem 265. *[Problem 16-4] If M is an oriented compact smooth manifold with boundary, then there does not exist a retraction of M onto its boundary.*

Proof. Suppose that there exists such a retraction $r : M \to \partial M$. By Theorem 6.26, we can assume that r is smooth. Let $\iota : \partial M \hookrightarrow M$ be the inclusion map. Let ω be an orientation form for ∂M. By Theorem 16.11 and Proposition 14.26,

$$\int_{\partial M} r^*\omega = \int_M d(r^*\omega) = \int_M r^*(d\omega) = 0$$

since ω is a top-degree form. By definition,

$$\int_{\partial M} r^*\omega = \int_{\partial M} \iota^* r^*\omega = \int_{\partial M} (r \circ \iota)^*\omega = \int_{\partial M} \omega,$$

so

$$\int_{\partial M} \omega = 0.$$

This contradicts Proposition 16.6(c). ∎

Theorem 266. *[Problem 16-5] Suppose M and N are oriented, compact, connected, smooth manifolds, and $F, G : M \to N$ are homotopic diffeomorphisms. Then F and G are either both orientation-preserving or both orientation-reversing.*

Proof. Theorem 253 shows that F and G are individually orientation-preserving or orientation-reversing. By Theorem 6.29, there is a smooth homotopy $H : M \times I \to N$ from F to G. Let ω be a positively oriented orientation form for N. By Theorem 16.11,

$$\int_{\partial(M\times I)} H^*\omega = \int_{M\times I} d(H^*\omega) = \int_{M\times I} H^*(d\omega) = 0.$$

But

$$\int_{\partial(M\times I)} H^*\omega = \int_M F^*\omega - \int_M G^*\omega,$$

so

$$\int_M F^*\omega = \int_M G^*\omega.$$

The result follows from Proposition 16.6(d). ∎

Theorem 267. *[Problem 16-6] The following are equivalent:*

1. *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.*
2. *There exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.*
3. *The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.*
4. *The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.*
5. *n is odd.*

Therefore, there exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.

Proof. Suppose X is a nowhere-vanishing vector field on $\mathbb{S}^n$. Then taking $V = X/|X|$ proves (1) $\Rightarrow$ (2). Suppose (2) holds. We can define a homotopy $H_1 : \mathbb{S}^n \times I \to \mathbb{S}^n$ from α to V by

$$H_1(x,t) = \frac{(1-t)\alpha(x) + tV(x)}{|(1-t)\alpha(x) + tV(x)|},$$

where the denominator is nonzero since $(1-t)(-x)+tV(x)=0$ contradicts the fact that $V(x)\perp x$. Similarly, we can define a homotopy H_2 from V to $\mathrm{Id}_{\mathbb{S}^n}$ by

$$H_2(x,t)=\frac{(1-t)V(x)+tx}{|(1-t)V(x)+tx|}.$$

This proves (2) $\Rightarrow$ (3). Theorem 266 proves (3) $\Rightarrow$ (4), and Theorem 254 proves (4) $\Rightarrow$ (5). Finally, Theorem 150 proves (5) $\Rightarrow$ (1). □

Theorem 268. *[Problem 16-9] Let ω be the $(n-1)$-form on $\mathbb{R}^n\setminus\{0\}$ defined by*

$$\omega=|x|^{-n}\sum_{i=1}^{n}(-1)^{i-1}x^i\,dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n.$$

(1) *$\iota^*_{\mathbb{S}^{n-1}}\omega$ is the Riemannian volume form of $\mathbb{S}^{n-1}$ with respect to the round metric and the standard orientation.*
(2) *ω is closed but not exact on $\mathbb{R}^n\setminus\{0\}$.*

Proof. The unit normal vector field for $\mathbb{S}^{n-1}$ with the standard orientation is given by $x\mapsto x$, so (1) follows from the formula in Lemma 14.13. For (2), we have

$$\begin{aligned}
d\omega&=\sum_{i=1}^{n}(-1)^{i-1}d\left(\frac{x^i}{|x|^n}\right)dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\\
&=\sum_{i=1}^{n}\left(\frac{|x|^n-n(x^i)^2\,|x|^{n-2}}{|x|^{2n}}\right)dx^1\wedge\cdots\wedge dx^n\\
&=\sum_{i=1}^{n}\left(\frac{1}{|x|^n}-\frac{n(x^i)^2}{|x|^{n+2}}\right)dx^1\wedge\cdots\wedge dx^n\\
&=0.
\end{aligned}$$

If ω is exact then

$$\int_{\mathbb{S}^{n-1}}\omega=0$$

by Corollary 16.13, which is a contradiction. □

Example 269. [Problem 16-10] Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(r-2)^2+z^2=1$ around the z-axis, with its induced Riemannian metric and with the orientation determined by the outward unit normal.

(1) Compute the surface area of D.
(2) Compute the integral over D of the function $f(x,y,z)=z^2+1$.
(3) Compute the integral over D of the 2-form $\omega=z\,dx\wedge dy$.

Let $\iota : D \hookrightarrow \mathbb{R}^3$ be the inclusion map. Let $U = (0, 2\pi) \times (0, 2\pi)$ and define an orientation-preserving diffeomorphism $F : U \to D$ by

$$F(\theta, \varphi) = ((2 + \cos\theta)\cos\varphi, (2 + \cos\theta)\sin\varphi, \sin\theta).$$

We have

$$\begin{aligned}
dx &= -\sin\theta\cos\varphi\, d\theta - (2 + \cos\theta)\sin\varphi\, d\varphi, \\
dy &= -\sin\theta\sin\varphi\, d\theta + (2 + \cos\theta)\cos\varphi\, d\varphi, \\
dz &= \cos\theta\, d\theta
\end{aligned}$$

and

$$\begin{aligned}
F^*\overline{g} &= dx^2 + dy^2 + dz^2 \\
&= \sin^2\theta\cos^2\varphi\, d\theta^2 + (2 + \cos\theta)^2\sin^2\varphi\, d\varphi^2 \\
&\qquad + 2\sin\theta\sin\varphi\cos\varphi(2 + \cos\theta)\, d\theta\, d\varphi \\
&\qquad + \sin^2\theta\sin^2\varphi\, d\theta^2 + (2 + \cos\theta)^2\cos^2\varphi\, d\varphi^2 \\
&\qquad - 2\sin\theta\sin\varphi\cos\varphi(2 + \cos\theta)\, d\theta\, d\varphi \\
&\qquad + \cos^2\theta\, d\theta^2 \\
&= d\theta^2 + (2 + \cos\theta)^2\, d\varphi^2.
\end{aligned}$$

For (1), we have

$$\begin{aligned}
\int_D \omega_{\iota^*\overline{g}} &= \int_{(\theta,\varphi)\in U} \sqrt{(2 + \cos\theta)^2} \\
&= 2\pi \int_0^{2\pi} (2 + \cos\theta)\, d\theta \\
&= 8\pi^2.
\end{aligned}$$

For (2), we have

$$\begin{aligned}
\int_D f\omega_{\iota^*\overline{g}} &= \int_{(\theta,\varphi)\in U} (\sin^2\theta + 1)\sqrt{(2 + \cos\theta)^2} \\
&= 2\pi \int_0^{2\pi} (\sin^2\theta + 1)(2 + \cos\theta)\, d\theta \\
&= 12\pi^2.
\end{aligned}$$

For (3), we have

$$\begin{aligned}
dx \wedge dy &= (-\sin\theta\cos\varphi\, d\theta - (2 + \cos\theta)\sin\varphi\, d\varphi) \\
&\qquad \wedge (-\sin\theta\sin\varphi\, d\theta + (2 + \cos\theta)\cos\varphi\, d\varphi) \\
&= -(2 + \cos\theta)\sin\theta\, d\theta \wedge d\varphi
\end{aligned}$$

so that

$$\begin{aligned}\int_D z\,dx\wedge dy &= -\int_D (2+\cos\theta)\sin^2\theta\,d\theta\wedge d\varphi\\ &= -2\pi\int_0^{2\pi}(2+\cos\theta)\sin^2\theta\,d\theta\\ &= -4\pi^2.\end{aligned}$$

Theorem 270. *[Problem 16-11] Let (M,g) be a Riemannian n-manifold with or without boundary. In any smooth local coordinates (x^i) we have*

$$\operatorname{div}\left(X^i\frac{\partial}{\partial x^i}\right)=\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det g}\right),$$

where $\det g=\det(g_{kl})$ is the determinant of the component matrix of g in these coordinates.

Proof. We have

$$\omega_g=\sqrt{\det g}dx^1\wedge\cdots\wedge dx^n$$

by Proposition 15.31, so

$$\beta\left(X^i\frac{\partial}{\partial x^i}\right)=\sum_{i=1}^n(-1)^{i-1}X^i\sqrt{\det g}\,dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n.$$

Then

$$d\left(\beta\left(X^i\frac{\partial}{\partial x^i}\right)\right)=\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det g}\right)\,dx^1\wedge\cdots\wedge dx^n$$

and

$$\begin{aligned}\operatorname{div}\left(X^i\frac{\partial}{\partial x^i}\right)&=*^{-1}\left(d\left(\beta\left(X^i\frac{\partial}{\partial x^i}\right)\right)\right)\\ &=\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det g}\right).\end{aligned}$$

□

Theorem 271. *[Problem 16-12] Let (M,g) be a compact Riemannian manifold with boundary, let $\widetilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.*

(1) *The divergence operator satisfies the following product rule for $f\in C^\infty(M)$, $X\in\mathfrak{X}(M)$:*

$$\operatorname{div}(fX)=f\operatorname{div}X+\langle\operatorname{grad}f,X\rangle_g.$$

(2) *We have the following "integration by parts" formula:*

$$\int_M \langle \operatorname{grad} f, X\rangle_g \, dV_g = \int_{\partial M} f \langle X, N\rangle_g \, dV_{\widetilde{g}} - \int_M (f \operatorname{div} X)\, dV_g.$$

Proof. Theorem 270 shows that in coordinates, we have

$$\begin{aligned}
\operatorname{div}(fX) &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(fX^i\sqrt{\det g}\right) \\
&= \frac{1}{\sqrt{\det g}}\left(f\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det g}\right) + \frac{\partial f}{\partial x^i}X^i\sqrt{\det g}\right) \\
&= f\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(X^i\sqrt{\det g}\right) + \frac{\partial f}{\partial x^i}X^i \\
&= f \operatorname{div} X + \langle \operatorname{grad} f, X\rangle_g .
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_M \langle \operatorname{grad} f, X\rangle_g \, dV_g &= \int_M \operatorname{div}(fX)\, dV_g - \int_M f \operatorname{div} X \, dV_g \\
&= \int_{\partial M} f \langle X, N\rangle_g \, dV_{\widetilde{g}} - \int_M f \operatorname{div} X \, dV_g
\end{aligned}$$

by Theorem 16.32. □

Theorem 272. *[Problem 16-13] Let (M, g) be a Riemannian n-manifold with or without boundary. The linear operator $\triangle : C^\infty(M) \to C^\infty(M)$ defined by $\triangle u = -\operatorname{div}(\operatorname{grad} u)$ is called the* ***(geometric) Laplacian****. The Laplacian is given in any smooth local coordinates by*

$$\triangle u = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\frac{\partial u}{\partial x^j}\right).$$

On $\mathbb{R}^n$ with the Euclidean metric and standard coordinates,

$$\triangle u = -\sum_{i=1}^n \frac{\partial^2 u}{(\partial x^i)^2}.$$

Proof. Obvious from Theorem 270. □

Theorem 273. *[Problem 16-14] Let (M, g) be a Riemannian n-manifold with or without boundary. A function $u \in C^\infty(M)$ is said to be* ***harmonic*** *if $\triangle u = 0$.*

(1) *If M is compact, then*

$$\int_M u \triangle v \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} uNv \, dV_{\widetilde{g}},$$

$$\int_M (u\triangle v - v\triangle u)\, dV_g = \int_{\partial M} (vNu - uNv) dV_{\widetilde{g}},$$

*where N and $\widetilde{g}$ are as in Theorem 271. These are known as **Green's identities**.*

(2) *If M is compact and connected and $\partial M = \emptyset$, the only harmonic functions on M are the constants.*

(3) *If M is compact and connected, $\partial M \neq \emptyset$, and u, v are harmonic functions on M whose restrictions to ∂M agree, then $u = v$.*

Proof. (1) follows by putting $f = u$ and $X = \operatorname{grad} v$ in Theorem 271. For (2), if u is harmonic then

$$\int_M |\operatorname{grad} u|_g^2 \, dV_g = \int_M u\triangle u \, dV_g = 0$$

by part (1). Therefore $\operatorname{grad} u = 0$, and u is constant since M is connected. For (3), we have

$$\int_M |\operatorname{grad}(u - v)|_g^2 \, dV_g = \int_{\partial M} (u - v)N(u - v)\, dV_{\widetilde{g}} = 0$$

by part (1). Therefore $\operatorname{grad}(u - v) = 0$, and $u - v$ is constant since M is connected. Since u and v agree on ∂M, we have $u = v$. □

Theorem 274. *[Problem 16-15] Let (M, g) be a compact connected Riemannian manifold without boundary, and let $\triangle$ be its geometric Laplacian. A real number λ is called an eigenvalue of $\triangle$ if there exists a smooth real-valued function u on M, not identically zero, such that $\triangle u = \lambda u$. In this case, u is called an **eigenfunction** corresponding to λ.*

(1) 0 *is an eigenvalue of $\triangle$, and all other eigenvalues are strictly positive.*

(2) *If u and v are eigenfunctions corresponding to distinct eigenvalues, then $\int_M uv \, dV_g = 0$.*

Proof. Any constant function is an eigenfunction with eigenvalue 0. If $\triangle u = \lambda u$ and u is not constant then Theorem 273 shows that

$$0 < \int_M |\operatorname{grad} u|_g^2 \, dV_g = \int_M u\triangle u \, dV_g = \lambda \int_M u^2 \, dV_g,$$

so $\lambda > 0$. Suppose $\triangle u = \lambda u$ and $\triangle v = \mu v$ where $\lambda \neq \mu$. By Theorem 273,

$$0 = \int_M (u\triangle v - v\triangle u)\, dV_g = (\mu - \lambda) \int_M uv \, dV_g,$$

so

$$\int_M uv \, dV_g = 0$$

since $\mu - \lambda \neq 0$. □

Theorem 275. *[Problem 16-16] Let M be a compact connected Riemannian n-manifold with nonempty boundary. A number $\lambda \in \mathbb{R}$ is called a* ***Dirichlet eigenvalue for*** *M if there exists a smooth real-valued function u on M, not identically zero, such that $\triangle u = \lambda u$ and $u|_{\partial M} = 0$. Similarly, λ is called a* ***Neumann eigenvalue*** *if there exists such a u satisfying $\triangle u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.*

(1) *Every Dirichlet eigenvalue is strictly positive.*
(2) 0 *is a Neumann eigenvalue, and all other Neumann eigenvalues are strictly positive.*

Proof. As in Theorem 274. □

Theorem 276. *[Problem 16-17] Suppose M is a compact connected Riemannian n-manifold with nonempty boundary. A function $u \in C^\infty(M)$ is harmonic if and only if it minimizes $\int_M |\operatorname{grad} u|_g^2 \, dV_g$ among all smooth functions with the same boundary values.*

Proof. Suppose $f \in C^\infty(M)$ vanishes on ∂M. For all ε, we have

$$\begin{aligned}\int_M |\operatorname{grad}(u+\varepsilon f)|_g^2 \, dV_g = \int_M |\operatorname{grad} u|_g^2 \, dV_g + 2\varepsilon \int_M \langle \operatorname{grad} u, \operatorname{grad} f\rangle_g \, dV_g \\ + \varepsilon^2 \int_M |\operatorname{grad} f|_g^2 \, dV_g.\end{aligned}$$

Applying Theorem 271, we have

$$\begin{aligned}\int_M |\operatorname{grad}(u+\varepsilon f)|_g^2 \, dV_g = \int_M |\operatorname{grad} u|_g^2 \, dV_g + 2\varepsilon \int_M (f \triangle u) \, dV_g \\ + \varepsilon^2 \int_M |\operatorname{grad} f|_g^2 \, dV_g.\end{aligned}$$

If u is harmonic then

$$\int_M |\operatorname{grad}(u+f)|_g^2 \, dV_g = \int_M |\operatorname{grad} u|_g^2 \, dV_g + \int_M |\operatorname{grad} f|_g^2 \, dV_g,$$

and it is clear that u minimizes $\int_M |\operatorname{grad} u|_g^2 \, dV_g$ among all smooth functions with the same boundary values. Conversely, suppose that u minimizes $\int_M |\operatorname{grad} u|_g^2 \, dV_g$. Then

$$\int_M |\operatorname{grad}(u+\varepsilon f)|_g^2 \, dV_g$$

has a global minimum at $\varepsilon = 0$, and its derivative (with respect to ε)

$$2\int_M (f \triangle u) \, dV_g + 2\varepsilon \int_M |\operatorname{grad} f|_g^2 \, dV_g$$

must be zero at $\varepsilon = 0$. That is,

$$\int_M (f \triangle u)\, dV_g = 0$$

for all $f \in C^\infty(M)$ such that $f|_{\partial M} = 0$. Therefore $\triangle u = 0$. $\square$

Theorem 277. *[Problem 16-18] Let (M, g) be an oriented Riemannian n-manifold.*

(1) *For each $k = 1, \dots, n$, g determines a unique inner product on $\Lambda^k(T_p^*M)$ (denoted by $\langle \cdot, \cdot \rangle_g$, just like the inner product on T_pM) satisfying*

$$\left\langle \omega^1 \wedge \cdots \wedge \omega^k, \eta^1 \wedge \cdots \wedge \eta^k \right\rangle_g = \det \left(\left\langle (\omega^i)^\#, (\eta^j)^\# \right\rangle_g \right)$$

whenever $\omega^1, \dots, \omega^k, \eta^1, \dots, \eta^k$ are covectors at p.

(2) *The Riemannian volume form dV_g is the unique positively oriented n-form that has unit norm with respect to this inner product.*

(3) *For each $k = 0, \dots, n$, there is a unique smooth bundle homomorphism $* : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ satisfying*

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_g \, dV_g$$

for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the ***Hodge star operator****.*

(4) *$* : \Lambda^0 T^*M \to \Lambda^n T^*M$ is given by $*f = f\, dV_g$.*

(5) *$** \omega = (-1)^{k(n-k)} \omega$ if ω is a k-form.*

Proof. For (1), any such inner product must satisfy

$$\begin{aligned}
\left\langle \varepsilon^I, \varepsilon^J \right\rangle_g &= \det \begin{bmatrix} \left\langle (\varepsilon^{i_1})^\#, (\varepsilon^{j_1})^\# \right\rangle_g & \cdots & \left\langle (\varepsilon^{i_1})^\#, (\varepsilon^{j_k})^\# \right\rangle_g \\ \vdots & \ddots & \vdots \\ \left\langle (\varepsilon^{i_k})^\#, (\varepsilon^{j_1})^\# \right\rangle_g & \cdots & \left\langle (\varepsilon^{i_k})^\#, (\varepsilon^{j_k})^\# \right\rangle_g \end{bmatrix} \\
&= \det \begin{bmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{bmatrix} \\
&= \delta^I_J \qquad (*)
\end{aligned}$$

for all increasing multi-indices I and J, whenever (ε^i) is the coframe dual to a local orthonormal frame (E_i). (Here (g^{ij}) is the inverse of $g_{ij}(p)$, which is the matrix of g in these coordinates.) This proves that the inner product is uniquely determined. To prove existence, we will define $\langle \cdot, \cdot \rangle_g$ by (*) and extend bilinearly. To check that the definition is independent of the choice of orthonormal frame, let $(\widetilde{\varepsilon}^i)$ be another coframe dual to a local orthonormal frame $(\widetilde{E}_i)$. Then

$$\widetilde{\varepsilon}^I = \sideset{}{'}\sum_L \alpha_L \varepsilon^L \quad \text{and} \quad \widetilde{\varepsilon}^J = \sideset{}{'}\sum_M \beta_M \varepsilon^M$$

where $\alpha_L = \widetilde{\varepsilon}^I(E_{l_1}, \ldots, E_{l_k})$ and $\beta_M = \widetilde{\varepsilon}^J(E_{m_1}, \ldots, E_{m_k})$, so

$$\begin{aligned}
\left\langle \widetilde{\varepsilon}^I, \widetilde{\varepsilon}^J \right\rangle_g &= \sideset{}{'}\sum_L \sideset{}{'}\sum_M \alpha_L \beta_M \left\langle \varepsilon^L, \varepsilon^M \right\rangle_g \\
&= \sideset{}{'}\sum_L \sideset{}{'}\sum_M \alpha_L \beta_M \delta^L_M \\
&= \sideset{}{'}\sum_L \alpha_L \beta_L \\
&= \sideset{}{'}\sum_L \widetilde{\varepsilon}^I(E_{l_1}, \ldots, E_{l_k}) \widetilde{\varepsilon}^J(E_{l_1}, \ldots, E_{l_k}).
\end{aligned}$$

Since $E_i = A_i^j \widetilde{E}_j$ for some orthonormal matrix (A_i^j), we have

$$\begin{aligned}
\left\langle \widetilde{\varepsilon}^I, \widetilde{\varepsilon}^J \right\rangle_g &= \sideset{}{'}\sum_L \sum_P A_{l_1}^{p_1} \cdots A_{l_k}^{p_k} \delta^I_P \sum_Q A_{l_1}^{q_1} \cdots A_{l_k}^{q_k} \delta^J_Q \\
&= \sideset{}{'}\sum_L \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) A_{l_1}^{i_{\sigma(1)}} \cdots A_{l_k}^{i_{\sigma(k)}} \sum_{\tau \in S_k} (\operatorname{sgn} \tau) A_{l_1}^{j_{\tau(1)}} \cdots A_{l_k}^{j_{\tau(k)}} \\
&= ???????????????
\end{aligned}$$

For (2), if ω is any positively oriented n-form then $\omega_p = c(dV_g)_p$ for some $c > 0$, and

$$\begin{aligned}
\left\langle \omega_p, \omega_p \right\rangle_g &= c^2 \left\langle dV_g, dV_g \right\rangle_g \\
&= c^2 \left\langle \varepsilon^1 \wedge \cdots \wedge \varepsilon^n, \varepsilon^1 \wedge \cdots \wedge \varepsilon^n \right\rangle_g \\
&= c^2
\end{aligned}$$

whenever (ε^i) is the coframe dual to a local orthonormal frame, by Proposition 15.29. If ω has unit norm then it is clear that $c = 1$. □

Example 278. [Problem 16-19] Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric and the standard orientation.

(1) Calculate $*dx^i$ for $i = 1, \ldots, n$.
(2) Calculate $*(dx^i \wedge dx^j)$ in the case $n = 4$.

TODO

Theorem 279. *[Problem 16-20] Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be **self-dual** if $*\omega = \omega$, and **anti-self-dual** if $*\omega = -\omega$.*

1. *Every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.*
2. *On $M = \mathbb{R}^4$ with the Euclidean metric, the self-dual and anti-self-dual forms in standard coordinates are given by:*

$$??????????$$

Proof. TODO □

Theorem 280. *[Problem 16-21] Let (M, g) be an oriented Riemannian manifold and $X \in \mathfrak{X}(M)$. Then*

$$X \lrcorner dV_g = *X^\flat,$$
$$\operatorname{div} X = *d * X^\flat,$$

and, when $\dim M = 3$,

$$\operatorname{curl} X = (*dX^\flat)^\sharp.$$

Proof. TODO □

Theorem 281. *[Problem 16-22] Let (M, g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Theorem 277. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \Omega^0(M)$.*

1. *$d^* \circ d^* = 0$.*
2. *The formula*

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

   defines an inner product on $\Omega^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Theorem 277.
3. *$(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$.*

Proof. TODO □

Theorem 282. *[Exercise 17.37] Suppose M, N, and P are compact, connected, oriented, smooth n-manifolds.*

(1) *If $F : M \to N$ and $G : N \to P$ are both smooth maps, then $\deg(G \circ F) = (\deg G)(\deg F)$.*
(2) *If $F : M \to N$ is a diffeomorphism, then $\deg F = +1$ if F is orientation-preserving and -1 if it is orientation-reversing.*
(3) *If two smooth maps $F_0, F_1 : M \to N$ are homotopic, then they have the same degree.*

Proof. For every smooth n-form ω on P, we have

$$\begin{aligned}\int_M (G \circ F)^* \omega &= \int_M F^* G^* \omega \\ &= (\deg F) \int_N G^* \omega \\ &= (\deg F)(\deg G) \int_P \omega.\end{aligned}$$

This proves (1). Part (2) follows from Proposition 16.6(d). If F_0 and F_1 are homotopic then Proposition 17.10 shows that the induced cohomology maps F_0^* and F_1^* are equal. For any smooth n-form ω on N, we have $F_0^*\omega - F_1^*\omega = d\eta$ for some smooth $(n-1)$-form η. Then

$$\int_M F_0^* \omega - \int_M F_1^* \omega = \int_M d\eta = 0.$$

□

Theorem 283. *[Problem 17-1] Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ be closed forms. The de Rham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map $\smile : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M)$, called the **cup product**, given by $[\omega] \smile [\eta] = [\omega \wedge \eta]$.*

Proof. TODO □

References

[1] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2nd edition, 2011.