Chapter 2. Basic Topology

Theorem 1. [Exercise 2.9(d)] For any set $E$, $(E^\circ)^c = \overline{E}^c$.

Proof. Suppose $x \notin \overline{E}^c = E^c \cup (E^\circ)'$, i.e. $x \in E$ and $x \notin (E^\circ)'$. Since $x$ is not a limit point of $E^c$ and $x \notin E^c$, there exists a neighborhood $N$ of $x$ such that $N \cap E^c$ is empty, i.e. $N \subseteq E$. This means $x \in E^\circ$. Then $x \in (E^\circ)^c = x \notin \overline{E}^c$, which shows that $(E^\circ)^c \subseteq \overline{E}^c$.

Suppose that $x \in \overline{E}^c = E^c \cup (E^\circ)'$, i.e. $x \notin E$ or $x$ is a limit point of $E^c$. If $x \notin E$ then $x \notin E^c$, which means $x \in (E^\circ)^c$. If $x$ is a limit point of $E^c$ then for any neighborhood $N$ of $x$ there exists a $y \neq x$ in $N$ such that $y \in E^c \Rightarrow y \notin E$. This shows that $x$ cannot be an interior point of $E$, so $x \in (E^\circ)^c$. Thus $\overline{E}^c = (E^\circ)^c$. □

Theorem 2. [Exercise 2.19(b)] If $A$ and $B$ are disjoint open sets, then they are separated.

Proof. We have $A \cap \overline{B} = A \cap (B \cup B') = A \cap B'$ since $A \cap B$ is empty. Suppose that there exists a $x \in A$ that is a limit point of $B$. Since $A$ is open, there exists a neighborhood $N$ of $x$ such that $N \subseteq A$. Since $x$ is a limit point of $B$, there exists a $y \in N$ such that $y \in B$. But then $y \in A$; this is a contradiction for $A$ and $B$ are disjoint. Therefore $A \cap B'$ is empty, and $A \cap \overline{B} = \emptyset$. Similarly, $B \cap \overline{A}$ is empty. This shows that $A$ and $B$ are separated. □

Theorem 3. [Exercise 2.21] Let $A$ and $B$ be separated subsets of some $\mathbb{R}^k$, suppose $a \in A$, $b \in B$, and define

$$ p(t) = (1 - t) a + t b $$

for $t \in \mathbb{R}^1$. Put $A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$. Then:

1. $A_0$ and $B_0$ are separated subsets of $\mathbb{R}^1$.
2. There exists a $t_0 \in (0, 1)$ such that $p(t_0) \notin A \cup B$.
3. Every convex subset of $\mathbb{R}^k$ is connected.

Proof. Let $x \in A_0$ so that $p(x) \in A$. Since $A$ and $B$ are separated, $p(x)$ is not a limit point of $B$ and $p(x) \notin B$. So there exists a neighborhood $N$ of $p(x)$ such that $N \cap B$ is empty. Consider $N_0 = p^{-1}(N)$, which is a neighborhood of $x$. For every $y \in N_0$ we have $p(y) \in N$ which means $p(y) \notin B$. But then $y \notin B_0$, so $x$ cannot be a limit point of $B_0$. This shows that $A_0 \cap B_0$ is empty. Similarly, $B_0 \cap A_0$ is empty. Hence $A_0$ and $B_0$ are separated.

We know that $A_0 \cup B_0 \subseteq (0, 1)$. Suppose that $A_0 \cup B_0 = (0, 1)$. Then $(0, 1)$ is the union of two separated sets by part (1), implying that it is disconnected. This is a
contradiction, so $A_0 \cup B_0$ is a proper subset of $(0,1)$ and there exists a $t_0 \in (0,1)$ such that $t_0 \notin A_0$ and $t_0 \notin B_0$, i.e. $p(t_0) \notin A \cup B$.

Let $C$ be a convex subset of $\mathbb{R}^k$ and suppose that $C = A \cup B$ where $A$ and $B$ are separated. Choose some $a \in A$ and $b \in B$. Then there exists a $t_0 \in (0,1)$ such that $(1 - t_0) a + t_0 b \notin C$ by statement (2). This contradicts the fact that $C$ is a convex set. Hence $C$ must be connected.

\[ \text{Lemma 6.} \]

Proof. Let $X$ be a separable metric space and let $A$ be a countable dense subset of $X$. Let $B = \{ V_{\alpha, r} \}$ be the collection of all neighborhoods $N_{\alpha}(Y)$ where $\alpha \in A$ and $r \in \mathbb{Q}$. $B$ is countable since $Y \times \mathbb{Q}$ is countable; we want to show that $B$ is a base for $X$. Let $E$ be an open set in $X$. For every $x \in E$, there exists a neighborhood $N$ of $x$ with radius $r$ such that $N \subseteq E$. Let $r_1$ be some positive rational number less than $r/2$ and let $N_1 = N_{r_1}(x)$. Since $x$ is a limit point of $Y$, there exists a $y \in N_1$ such that $y \in Y$. Now let $V = N_{r_1}(y)$; since $d(x, y) < r_1$, $x \in V$. Also $V \subseteq N \subseteq E$, since for every $v \in V$, $d(v, x) \leq d(v, y) + d(y, x) < 2r_1 < r$. Since $y \in Y$ and $r_1 \in \mathbb{Q}$, $V \in B$. This shows that $B$ is a countable base for $X$. \[ \square \]

\[ \text{Theorem 4.} \] [Exercise 2.23] Every separable metric space has a countable base.

Proof. Fix $\delta > 0$ and choose $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Suppose that this process does not terminate after a finite number of steps. Then we have an infinite set $S = \{ x_1, x_2, \ldots \}$ in which $d(x_i, x_j) \geq \delta$ for every $j \neq i$. Suppose that $x_0$ is a limit point of $S$. Then there are an infinite number of elements $x_i \in S$ such that $d(x_0, x_i) < \delta/2$. But if $x_i, x_j$ are two such elements, $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \delta$, which is a contradiction. Hence $S$ cannot have any limit points. This contradicts the assumption that every infinite subset has a limit point, so the process must terminate after a finite number of steps. Let $S_\delta = \{ x_1, x_2, \ldots \}$ be the set of points found by this process for some $\delta$.

The union $C = N_\delta(x_1) \cup N_\delta(x_2) \cup \cdots$ covers $X$ for if $x \in X \setminus C$, then $x$ would have been added to $S_\delta$. Let $D = \bigcup_{n=1}^\infty S_{1/n}$; we want to show that $D$ is a countable dense subset of $X$. That $D$ is countable is clear since each $S_{1/n}$ is finite. Let $x \in X$ and let $N$ be a neighborhood of $x$ with radius $r$. Let $n$ be a positive integer such that $n > 1/r$. There exists some $S_{1/n} \subseteq D$ and some $s \in S_{1/n}$ such that $N_{1/n}(s)$ contains $x$, since $\bigcup_{s \in S_{1/n}} N_{1/n}(s)$ covers $X$. Now $d(s, x) < 1/n < r$, so $s \in N$. Therefore $x$ is a limit point of $D$. This proves that $X$ is separable. \[ \square \]

\[ \text{Theorem 5.} \] [Exercise 2.24] If $X$ is a metric space in which every infinite subset has a limit point, then $X$ is separable.

\[ \text{Lemma 6.} \] Let $X$ be a metric space with a countable base. Then $X$ is separable.
**Proof.** Let \( V = \{V_1, V_2, \ldots \} \) be a countable base for \( X \). For every \( i \) choose an element \( x_i \in V_i \), and let \( D = \{x_1, x_2, \ldots \} \); \( D \) is countable since \( V \) is countable. Let \( x \in X \) and let \( N \) be a neighborhood of \( x \). Then \( N \) is the union of a subcollection of \( V \) and therefore contains some element from \( D \). This shows that \( x \) is a limit point of \( D \), and that \( D \) is dense in \( X \). \( \square \)

**Theorem 7.** [Exercise 2.25] Every compact metric space \( K \) has a countable base, and \( K \) is therefore separable.

**Proof.** Let \( B_n \) be the collection of all neighborhoods \( N_r(\alpha) \) with \( r = 1/n \) and \( \alpha \in K \). Since \( B_n \) is an open cover of \( K \) and \( K \) is compact, there exists a finite subcover
\[ C_n = \{V_1, V_2, \ldots, V_k\} \subset B_n \]
that covers \( K \). Let
\[ C = C_1 \cup C_2 \cup \cdots \; C \]
is countable since each \( C_i \) is countable. Let \( E \) be an open set in \( K \). For every \( x \in E \), there exists a neighborhood \( N \) of \( x \) with radius \( r \) such that \( N \subseteq E \). Let \( n \) be a positive integer such that \( n > 2/r \). There exists some neighborhood \( N_1 \in C_n \) centered at \( \alpha \) such that \( x \in N_1 \), since \( C_n \) covers \( K \). Also, \( N_1 \subseteq N \subseteq E \) since for every \( y \in N_1 \),
\[ d(x, y) \leq d(x, \alpha) + d(\alpha, y) < 1/n + 1/n < r. \]
This shows that \( C \) is a countable base for \( K \). Lemma 6 shows that \( K \) is separable. \( \square \)

**Theorem 8.** [Exercise 2.26] If \( X \) is a metric space in which every infinite subset has a limit point, then \( X \) is compact.

**Proof.** By Theorem 5 \( X \) is separable, and by Theorem 4 \( X \) has a countable base
\[ V = \{V_1, V_2, \ldots \} \]
Let \( \{G_\alpha\} \) be an open cover of \( X \). For every \( x \in X \), there is some open set \( G_\alpha \) such that \( x \in G_\alpha \). Since \( V \) is a base for \( X \), there exists a \( V_i \in V \) with \( x \in V_i \subseteq G_\alpha \). This means that there is a countable subcover \( \{G_1\} \) of \( X \) since each \( G_\alpha \) was associated with an element of \( V \). Suppose that no finite subcollection of \( \{G_i\} \) covers \( X \). For every positive integer \( n \), let \( F_n = (G_1 \cup \cdots \cup G_n)^c \). Since \( \{G_1, \ldots, G_n\} \) is a finite subcollection, each \( F_n \) is nonempty while \( \bigcap_{n=1}^\infty F_n = \bigcup_{n=1}^\infty G_i \) is empty since \( \{G_i\} \) covers \( X \).

Let \( E = \{f_1, f_2, \ldots \} \) be a set where each \( f_i \) is chosen from \( F_i \). Since \( E \) is an infinite subset of \( X \), \( E \) has a limit point \( x \). Suppose that \( x \notin F_i \) for some \( i \). Since \( F_i^c \) is open, there exists a neighborhood \( N \) of \( x \) with radius \( r \) such that \( N \cap F_i = \emptyset \). In fact, \( N \cap F_j = \emptyset \) for every \( j \geq i \) since \( F_1 \supseteq F_2 \supseteq \cdots \), and therefore \( N \cap E \) is finite. But \( x \) is a limit point of \( E \), so \( N \cap E \) must be infinite. This is a contradiction, and therefore \( x \in F_i \) for all \( i \). Then \( x \in \bigcap_{n=1}^\infty F_n \), but this is a contradiction for \( \bigcap_{n=1}^\infty F_n \) is empty. Thus there is a finite subcollection of \( \{G_i\} \) that covers \( X \), and \( X \) must be compact. \( \square \)

**Chapter 3. Numerical Sequences and Series**

**Theorem 9.** A sequence \( \{p_n\} \) converges to \( p \) if and only if every subsequence of \( \{p_n\} \) converges to \( p \).
Proof. Suppose that \( \{p_n\} \) converges to \( p \) and let \( \{p_{n_i}\} \) be a subsequence of \( \{p_n\} \). Let \( \varepsilon > 0 \) be given. Then there exists an integer \( N \) such that for every \( n \geq N \), \( d(p_n, p) < \varepsilon. \) Let \( N' \) be the smallest \( i \) such that \( n_i \geq N. \) Then for every \( i \geq N' \), \( d(p_{n_i}, p) < \varepsilon \). Therefore \( \{p_{n_i}\} \) converges to \( p \). Conversely, suppose that every subsequence of \( \{p_n\} \) converges to \( p \). \( \{p_n\} \) is a subsequence of itself, so it converges to \( p \).

**Theorem 10.** Let \( \{s_n\} \) and \( \{t_n\} \) be sequences in \( \mathbb{R} \). If \( s_n \leq t_n \) for \( n \geq N \) where \( N \) is some constant, if \( s_n \to s \), and if \( t_n \to t \), then \( s \leq t \).

**Proof.** Assume \( s \neq t \) so that \( |t - s| > 0 \), for otherwise we are done. Since \( s_n \to s \) and \( t_n \to t \), \( t_n - s_n \to t - s \). There exists a \( M \) such that for every \( m \geq M \), 
\[
|t_m - s_m - (t - s)| < |t - s|
\]
Whenever \( k \geq \max(M, N) \), both \( t_k - s_k \geq 0 \) and \( t_k - s_k - (t - s) < |t - s| \) hold. We know \( t - s > 0 \) for if \( t - s < 0 \), then \( t_k - s_k < 0 \) which is a contradiction.

**Theorem 11.** Let \( \{x_n\} \) and \( \{s_n\} \) be sequences in \( \mathbb{R} \). If \( 0 \leq x_n \leq s_n \) for \( n \geq N \) where \( N \) is some constant, and if \( s_n \to 0 \), then \( x_n \to 0 \).

**Proof.** Let \( \varepsilon > 0 \) be given. Since \( s_n \to 0 \), there exists a \( M \) such that for every \( n \geq M \), 
\[
|s_n| < \varepsilon.
\]
Let \( N' = \max(M, N) \); then for every \( n \geq N' \), 
\[
|x_n| \leq s_n \leq \varepsilon.
\]
Therefore \( x_n \to 0 \).

**Corollary 12.** Let \( \{x_n\}, \{s_n\}, \{s'_n\} \) be sequences in \( \mathbb{R} \). If \( s_n \leq x_n \leq s'_n \) for \( n \geq N \) where \( N \) is some constant, if \( s_n \to s \), and if \( s'_n \to s \), then \( x_n \to s \).

**Theorem 13.** Let \( \{s_n\}, \{t_n\} \) be sequences in a metric space. If \( s_n \to s \) and \( d(s_n, t_n) \to 0 \), then \( t_n \to s \).

**Proof.** Let \( \varepsilon > 0 \) be given. There exists a \( M \) such that \( d(s_n, t_n) < \varepsilon/2 \) whenever \( n \geq M \), and there exists a \( N \) such that \( d(s, s_n) < \varepsilon/2 \) whenever \( n \geq N \). Then for all \( n \geq \max(M, N) \) we have
\[
d(s, t_n) \leq d(s, s_n) + d(s_n, t_n) < \varepsilon.
\]

**Theorem 14.** ([Theorem 3.19]) If \( s_n \leq t_n \) for \( n \geq N \) where \( N \) is fixed, then
\[
\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \quad \text{and} \quad \liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n.
\]

**Proof.** Let \( E_1 \) be the set of subsequential limits of \( \{s_n\} \) and let \( E_2 \) be the set of subsequential limits of \( \{t_n\} \). Let \( L_1 = \limsup_{n \to \infty} s_n \) and \( L_2 = \limsup_{n \to \infty} t_n \). If \( L_1 = -\infty \) or \( L_2 = +\infty \), then there is nothing to prove. Otherwise, \( L_1 \in E_1 \) and there exists a subsequence \( \{s_{n_i}\} \) that converges to \( L_1 \). Similarly, some \( \{t_{n_i}'\} \) converges to \( L_2 \). Let \( m_1 \)
be the minimum $i$ such that $n_i \geq N$ and let $m_2$ be the minimum $i$ such that $n'_i \geq N$. Let $M = \max(m_1, m_2)$; then $s_{n_i} \leq t_{n'_i}$ for all $i \geq M$ since $s_n \leq t_n$ whenever $n \geq N$. Theorem 10 proves the required result. The case for lim inf is similar. □

**Lemma 15.** Let $S = \{s_n\}$ be a sequence in $\mathbb{R}$ and let $E$ be the set of subsequential limits of $\{s_n\}$. Then $\sup E \in (-\infty, +\infty)$ if and only if $S$ is bounded.

*Proof.* Suppose that $S$ is not bounded above, i.e. for every $x \in \mathbb{R}$ there exists a $s_i \in S$ such that $s_i > x$. Let $n_1 = 1$ and suppose that $n_1, \ldots, n_k$ have been chosen. Choose $n_{k+1}$ to be the smallest $i$ such that $i > n_k$ and $s_i > s_{n_k}$. Then the subsequence $\{s_{n_k}\}$ approaches $+\infty$ and hence $\sup E = +\infty$. Similarly, if $S$ is not bounded below then $\sup E = -\infty$. Conversely, if $\sup E = +\infty$ then there exists a subsequence $\{s_{n_k}\}$ such that for every $M$, $s_{n_k} \geq M + 1 > M$ for some $n_k$. The case for $\sup E = -\infty$ is similar. Hence $S$ is unbounded. □

**Theorem 16.** [Equivalence of $\limsup$ definitions.] Let $S = \{s_n\}$ be a sequence in $\mathbb{R}$, let $S_n = \{s_n, s_{n+1}, \ldots\}$ and let $E$ be the set of subsequential limits of $\{s_n\}$. Let $L \in [\liminf, \limsup]$. Then the following are equivalent:

1. $L = \limsup E$.
2. $L \in E$ and for every $x > L$ there is an integer $N$ such that $n \geq N$ implies $s_n < x$.
3. $L = \lim_{n \to \infty} \sup S_n$.

Furthermore, any $L$ with these properties is unique.

*Proof.* We will show that (1) $\iff$ (2) and (1) $\iff$ (3). Suppose that $L = \limsup E$ and let $x$ be a number with $x > L$. That $L \in \limsup E$ is clear. We can now assume that $L < +\infty$, for if $L = +\infty$ then there is no such $x$ greater than $L$. Suppose that $s_n \geq x$ for infinitely many values of $n$; this forms a subsequence of $\{s_n\}$ consisting of all $s_{n_i} \geq x$. Some subsequence of this subsequence converges to a value $y$, since $s_{n_i} \geq x$ and $\limsup E < +\infty$ implies that $\{s_{n_i}\}$ is bounded by Lemma 15. Then $L \geq y \geq x > L$, which is a contradiction. Conversely, suppose that (2) holds for $L$ and suppose that $L < \limsup E$. Then choose $x$ such that $L < x < \limsup E$, and there is an integer $N$ such that $n \geq N$ implies $s_n < x$. Every subsequence of $\{s_n\}$ must have a limit no greater than $x < \limsup E$ by Theorem 10 and this contradicts the fact that $\limsup E$ is the least upper bound. Therefore $L \geq \limsup E$, and since $L \in E$, $L = \limsup E$. This proves (1) $\iff$ (2).

Let $L = \limsup E$ so that (2) holds. Let $\varepsilon > 0$ be given. There exists an integer $N$ such that $n \geq N$ implies $s_n < L + \varepsilon/2$. Whenever $n \geq N$, $\sup S_n \leq L + \varepsilon/2$ so that $\sup S_n - L < \varepsilon$. Suppose that $\sup S_n < L$; we can choose $x$ such that $\sup S_n < x < L$. Since every $s_k$ with $k \geq n$ has $s_k < x$, every subsequence of $\{s_n\}$ must have a limit no greater than $x < \limsup E$ by Theorem 10. Since $L$ is the least upper bound of $E$,
which is a contradiction. Therefore \( 0 \leq \sup S_n - L < \varepsilon \), showing that \( \lim_{n \to \infty} \sup S_n = L \). This proves \((1) \Leftrightarrow (3)\). □

**Theorem 17.** [Exercise 3.5] For any two real sequences \( \{a_n\} \) and \( \{b_n\} \),

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,
\]

provided that the sum on the right is not of the form \( \infty - \infty \).

**Proof.** If \( \limsup_{n \to \infty} (a_n + b_n) = \pm \infty \) then we are done. Otherwise, let

\[
L = \limsup_{n \to \infty} (a_n + b_n),
L_1 = \limsup_{n \to \infty} a_n,
L_2 = \limsup_{n \to \infty} b_n.
\]

There is a subsequence \( \{c_n\} \) of \( \{a_n + b_n\} \) that converges to \( L \). For each \( n_i \), \( c_{n_i} = a_{n_i} + b_{n_i} \) for some subsequences \( \{a_{n_i}\}, \{b_{n_i}\} \) so that \( L = a + b \) if we let \( a \) be the limit of \( a_{n_i} \) and \( b \) be the limit of \( b_{n_i} \). Then \( L = a + b \leq L_1 + L_2 \), which proves the result. □

**Theorem 18.** [Exercise 3.7] If \( a_n \geq 0 \) for all \( n \) and \( \sum a_n \) converges, then \( \sum \sqrt{a_n} \) converges.

**Proof.** Let \( t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \); clearly \( t_n \geq 0 \) for all \( n \). Let \( b_k = 1/k \), and by the Cauchy-Schwarz inequality,

\[
\left( \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \right)^2 \leq \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k^2}
\]

\[
t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \leq \sqrt{ \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k^2} } \leq \sqrt{ab}
\]

where \( a = \lim_{n \to \infty} a_n \) and \( b = \lim_{n \to \infty} 1/n^2 \). Thus \( \{t_n\} \) must be a bounded sequence and hence \( \sum \sqrt{a_n} \) converges. □

**Theorem 19.** [Exercise 3.8] If \( \sum a_n \) converges and \( \{b_n\} \) is monotonic and bounded, then \( \sum a_n b_n \) converges.

**Proof.** Suppose that \( \{b_n\} \) is monotonically increasing and let \( B \) be the limit of \( \{b_n\} \) so that \( b_n \leq B \) for every \( n \). Let \( C = B \sum a_n - \sum a_n (B - b_n) \). Since \( B - b_n \to 0 \)
and \( \{B - b_n\} \) is monotonically decreasing, we can apply Theorem 3.42 to deduce that 
\[ \sum a_n (B - b_n) \]
converges. Then
\[
C = B \sum a_n - \sum a_n (B - b_n)
= \sum a_n b_n
\]
converges. The case for \( \{b_n\} \) being monotonically decreasing is similar.

\[\square\]

**Theorem 20.** [Exercise 3.10] If \( \sum a_n z^n \) is a power series where infinitely many coefficients are distinct from zero, then the radius of convergence is at most 1.

**Proof.** Suppose that the radius of convergence \( R > 1 \), i.e. \( \sum a_n \gamma^n \) converges for some \( 1 < \gamma < R \). By the root test, \( \limsup_{n \to \infty} \sqrt[n]{|a_n \gamma^n|} = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq 1 \), which means that \( \limsup_{n \to \infty} \sqrt[n]{|a_n|} = L \) where \( L < 1 \). There exists some subsequence \( S = \left\{ \sqrt[n]{|a_n|} \right\} \) that converges to \( L \), and the neighborhood \( N_{1-L}(L) \) contains infinitely many points \( a_k \) of \( S \) with \( 0 \leq \sqrt[n]{|a_k|} < 1 \). But then infinitely many points \( a_k \) have \( 0 \leq |a_k| < 1 \), and thus infinitely many points are zero since each \( a_k \) is an integer. This is a contradiction, so the radius of convergence must not be greater than 1.

\[\square\]

**Theorem 21.** [Exercise 3.11] Suppose that \( a_n > 0 \), \( s_n = a_1 + \cdots + a_n \) and that \( \sum a_n \) diverges. Then:

1. The series \( \sum \frac{a_n}{1 + a_n} \) diverges.
2. For all \( N, k \geq 1 \), \( \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} \) and \( \sum \frac{a_n}{s_n} \) diverges.
3. For all \( n \), \( \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n} \) and \( \sum \frac{a_n}{s_n^2} \) converges.
4. \( \sum \frac{a_n}{1 + na_n} \) sometimes converges and \( \sum \frac{a_n}{1 + n^2a_n} \) always converges.

**Proof.** Suppose that \( \sum \frac{a_n}{1 + a_n} \) converges. Then \( \lim_{n \to \infty} \frac{a_n}{1 + a_n} = 0 \), and \( \lim_{n \to \infty} a_n = 0 \) (this can be shown using an \( \varepsilon \) argument). There exists an integer \( N \) such that \( a_n < 1 \) whenever \( n \geq N \), and furthermore since \( \sum \frac{a_n}{1 + a_n} \) converges, for any \( \varepsilon > 0 \) there exists an integer \( M \) such that \( \sum_{k=m}^{n} \frac{a_k}{1 + a_k} < \varepsilon / 2 \) whenever \( n \geq m \geq M \). Therefore whenever
\[ n \geq m \geq \max(M, N), \]
\[ \varepsilon > 2 \sum_{k=m}^{n} \frac{a_k}{1 + a_k} \]
\[ > 2 \sum_{k=m}^{n} \frac{a_k}{1 + 1} \]
\[ > \sum_{k=m}^{n} a_k \]

and \( \sum a_n \) converges. This shows that \( \sum \frac{a_n}{1 + a_n} \) diverges if \( \sum a_n \) diverges.

For \( N, k \geq 1 \),
\[ \sum_{k=N+1}^{N+k} a_n = a_{N+1} + a_{N+2} + \cdots + a_{N+k} \]
\[ 1 - \frac{s_N}{s_{N+k}} = \frac{a_{N+1} + a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \]
\[ \leq \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+1}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \]

Suppose that \( \sum \frac{a_n}{s_n} \) converges. Then there exists a \( N \) such that whenever \( n+j \geq n \geq N \),
\[ 1 - \frac{s_n}{s_{n+j}} \leq \sum_{k=n}^{n+j} \frac{a_n}{s_n} < \frac{1}{2} \]
so that for all \( j \), \( 2s_n > s_{n+j} \). But \( \{s_n\} \) is not bounded since \( \sum a_n \) diverges, and there is some \( j \) such that \( s_{n+j} > 2s_n \). This is a contradiction, so \( \sum \frac{2n}{s_n} \) cannot converge.

For the third inequality,
\[ 1 < \frac{s_n}{s_{n-1}} \]
\[ a_n < \frac{s_n(s_n - s_{n-1})}{s_{n-1}} \]
\[ \frac{a_n}{s_n} < \frac{s_n - s_{n-1}}{s_n s_{n-1}} \]
\[ = \frac{1}{s_{n-1}} - \frac{1}{s_n} \]
For any $\varepsilon > 0$, there is some $N$ for which $s_{N-1} > \frac{1}{\varepsilon}$ since $\{s_n\}$ is not bounded. Then for all $n \geq m \geq N$,

$$\sum_{k=m}^{n} \frac{a_n}{s_n^2} < \sum_{k=m}^{n} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) < \frac{1}{s_{m-1}} - \frac{1}{s_n} < \frac{1}{s_{m-1}} - \frac{1}{s_n} < \varepsilon$$

since $\{s_n\}$ is monotonically increasing. Hence $\sum \frac{a_n}{s_n^2}$ converges.

The series $\sum \frac{a_n}{1+n^2a_n}$ may or may not converge. If $a_n = 1$ then the series does not converge, but if $a_n = [n = m^2]$ where $[\ldots]$ is the Iverson bracket, then the series converges. The series $\sum \frac{a_n}{1+n^2a_n}$ always converges since $\frac{a_n}{1+n^2a_n} = \frac{1}{a_n+n^2} < \sum \frac{1}{n^2}$ and the series on the right hand side converges.

**Theorem 22.** [Exercise 3.12] Suppose that $a_n > 0$ and that $\sum a_n$ converges. Let $r_n = \sum_{m=n}^{\infty} a_m$. Then:

1. If $m < n$ then $\frac{a_m}{r_n} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$, and $\sum \frac{a_n}{r_n}$ diverges.
2. For any $n$, $\frac{a_n}{\sqrt{r_n}} < 2 (\sqrt{r_n} - \sqrt{r_{n+1}})$, and $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

**Proof.** If $m < n$ then

$$r_m - r_n < a_m + a_{m+1} + \ldots + a_n$$

$$1 - \frac{r_n}{r_m} < \frac{a_m}{r_m} + \frac{a_{m+1}}{r_m} + \ldots + \frac{a_n}{r_n} < \frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \ldots + \frac{a_n}{r_n}.$$ 

Suppose that $\sum \frac{a_n}{r_n}$ converges. Then there exists an integer $N$ such that for all $n \geq m \geq N$,

$$1 - \frac{r_n}{r_m} < \sum_{k=m}^{n} \frac{a_k}{r_k} < \frac{1}{2}$$

so that for all $n > m$, $2r_n > r_m$. Since $\sum a_n$ converges, $a_n \to 0$ which means $r_n \to 0$. Hence we can find an integer $n$ such that $r_n < r_m/2$, which is a contradiction. This shows that $\sum \frac{a_n}{r_n}$ does not converge.
To prove the second inequality,

\[4r_n (r_n - a_n) < 4r_n^2 - 4a_n r_n + a_n^2 = (2r_n - a_n)^2\]

\[2\sqrt{r_n} \sqrt{r_n - a_n} < 2r_n - a_n\]

\[a_n < 2 \left(r_n - \sqrt{r_n \sqrt{r_n - a_n}}\right)\]

\[\frac{a_n}{\sqrt{r_n}} < 2 \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)\]

For any \(\varepsilon > 0\), there exists some integer \(N\) such that \(r_N < \left(\frac{\varepsilon}{2}\right)^2\) since \(r_n \to 0\). Then for all \(n \geq m \geq N\),

\[\sum_{k=m}^{n} \frac{a_k}{\sqrt{r_k}} < 2 \sum_{k=m}^{n} \left(\sqrt{r_k} - \sqrt{r_{k+1}}\right)\]

\[< 2 \left(\sqrt{r_m} - \sqrt{r_{n+1}}\right)\]

\[< \varepsilon\]

since \(\{r_n\}\) is monotonically decreasing. Hence \(\sum \frac{a_n}{\sqrt{r_n}}\) converges. \(\square\)


**Proof.** Let \(\sum a_n\) and \(\sum b_n\) be two absolutely convergent series; we have \(\sum |a_n| \leq M_1\) and \(\sum |b_n| \leq M_2\) for some \(M_1, M_2\). Let \(c_n = \sum_{k=0}^{n} a_k b_{n-k}\). For all \(n\),

\[\sum_{k=0}^{n} |c_k| = \sum_{k=0}^{n} \left| \sum_{j=0}^{k} a_j b_{k-j} \right|\]

\[\leq \sum_{k=0}^{n} \sum_{j=0}^{k} |a_j| |b_{k-j}|\]

\[= \sum_{0 \leq j \leq k \leq n} |a_j| |b_{k-j}|\]

\[\leq \sum_{0 \leq j, k \leq n} |a_j| |b_{n-j}|\]

\[= \left(\sum_{j=0}^{n} |a_j|\right) \left(\sum_{k=0}^{n} |b_k|\right)\]

\[\leq M_1 M_2\]
so that sequence of partial sums of $\sum |c_n|$ is bounded. Therefore $\sum c_n$ converges absolutely.

**Theorem 24.** [Exercise 3.20] Let $\{p_n\}$ be a Cauchy sequence in a metric space $X$ where some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Then the sequence $\{p_n\}$ converges to $p$.

**Proof.** Let $\varepsilon > 0$ be given. There exists some $N$ such that for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon/2$. Also, there exists some $K$ such that for all $k \geq K$, $d(p_{n_k}, p) < \varepsilon/2$. Let $j$ be the smallest integer such that $n_j \geq \max(N, n_K)$. Then for all $n \geq n_j$, $d(p_n, p) \leq d(p_n, p_{n_j}) + d(p_{n_j}, p) < \varepsilon$. This shows that $p_n \to p$. □

**Theorem 25.** [Exercise 3.21] If $\{E_n\}$ is a sequence of closed, nonempty and bounded sets in a complete metric space $X$, if $E_n \supseteq E_{n+1}$, and if $\lim_{n \to \infty} \text{diam } E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

**Proof.** Let $\{p_n\}$ be a sequence where each $p_i$ is a point chosen from $E_i$. Let $\varepsilon > 0$ be given. Since $\text{diam } E_n \to 0$, there exists some $N$ such that $\text{diam } E_n < \varepsilon$ whenever $n \geq N$. Then for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon$ since $p_m, p_n \in E_N$. This shows that $\{p_n\}$ is a Cauchy sequence, and since $X$ is complete, $\{p_n\}$ converges. Suppose that $p \not\in E_i$ for some $i$. Then $p \in E_i^c$ and since $E_i^c$ is open, there exists some neighborhood $N$ of $p$ with radius $r$ such that $N \cap E_i = \emptyset$. In fact, $N \cap E_j = \emptyset$ for every $j \geq i$ since $E_1 \supseteq E_2 \supseteq \cdots$. Since $\{p_n\}$ converges to $p$, there exists some $M$ such that $d(p_m, p) < r$ whenever $m \geq M$. Let $k = \max(i, M)$ and consider $p_k$; we have $p_k \in E_k$ but $p_k \not\in E_i$ since $k \geq M$, which means that $p_k \not\in E_i$ and $p_k \not\in E_k$. This is a contradiction, so $p \in E_i$ for all $i$, i.e. $\bigcap_{n=1}^{\infty} E_n$ is nonempty. Furthermore, since $\text{diam } E_n \to 0$, $\bigcap_{n=1}^{\infty} E_n$ must consist of exactly one point. □

**Theorem 26.** [Exercise 3.22, Baire’s theorem] If $X$ is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of $X$, then $\bigcap_{n=1}^{\infty} G_n$ is not empty.

**Proof.** Let $g_1$ be a point in $G_1$ and let $N_1$ be a neighborhood of $g_1$ wholly contained in $G_1$. Let $E_1$ be a neighborhood of $g_1$ such that $\overline{E_1} \subseteq N_1$. Having constructed $E_1, \ldots, E_n$ such that $E_1 \supseteq \cdots \supseteq E_n$ and $\overline{E}_{i+1} \subseteq E_i \subseteq G_i$ for each $i$, let $g_n$ be the center of $E_n$. Since $G_{n+1}$ is dense in $X$, $E_n$ contains a point $g_{n+1} \in G_{n+1}$. Let $E_{n+1}$ be a neighborhood of $g_{n+1}$ such that $\overline{E}_{n+1} \subseteq E_n$. We can continue this process to obtain a sequence $\overline{E}_1 \supseteq \overline{E}_2 \supseteq \cdots$. By Theorem 25, there is exactly one point $x \in \bigcap_{n=1}^{\infty} \overline{E}_n$. But we have $E_i \subseteq G_i$ for each $i$, which means that $x \in \bigcap_{n=1}^{\infty} G_n$ and therefore $\bigcap_{n=1}^{\infty} G_n$ is not empty. □
**Theorem 27.** [Exercise 3.23] Let \( \{p_n\} \) and \( \{q_n\} \) be Cauchy sequences in a metric space \( X \). Then the sequence \( \{d(p_n, q_n)\} \) converges.

*Proof.* Let \( \varepsilon > 0 \) be given. There exists, by taking a maximum, an integer \( N \) such that for all \( m, n \geq N \),

\[
d(p_m, p_n) < \frac{\varepsilon}{2} \quad \text{and} \quad d(q_m, q_n) < \frac{\varepsilon}{2}.
\]

Then

\[
d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, p_n) + d(q_m, q_n)
\]

and similarly,

\[
d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_m) + d(q_n, q_m).
\]

This shows that

\[
|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m) < \varepsilon
\]

which means that \( \{d(p_n, q_n)\} \) converges. \( \square \)

**Theorem 28.** [Exercise 3.24] Let \( X \) be a metric space.

1. Call two Cauchy sequences \( \{p_n\}, \{q_n\} \) in \( X \) equivalent if \( \lim_{n \to \infty} d(p_n, q_n) = 0 \). This is an equivalence relation.
2. Let \( X^* \) be the set of all equivalence classes obtained by the above equivalence relation. If \( P \in X^* \), \( Q \in X^* \), \( \{p_n\} \in P \), \( \{q_n\} \in Q \), define \( \triangle(P, Q) = \lim_{n \to \infty} d(p_n, q_n) \). The number \( \triangle(P, Q) \) is unchanged if \( \{p_n\} \) and \( \{q_n\} \) are replaced by equivalent sequences, and hence that \( \triangle \) is a distance function in \( X^* \).
3. The metric space \( X^* \) is complete.
4. For each \( p \in X \), there is a Cauchy sequence all of whose terms are \( p \); let \( P_p \) be the element of \( X^* \) which contains this sequence. Then \( \triangle(P_p, P_q) = d(p, q) \) for all \( p, q \in X \).
5. Let \( \varphi : X \to X^* \) be given by \( p \mapsto P_p \) where \( P_p \) is the element of \( X^* \) which contains a sequence with all terms equal to \( p \). Then \( \varphi(X) \) is dense in \( X^* \), and if \( X \) is complete, then \( \varphi(X) = X^* \).
6. The completion of \( Q \) is \( \mathbb{R} \).

*Proof.* It is obvious that that the relation is reflexive and symmetric. Let \( \{p_n\}, \{q_n\}, \{r_n\} \) be sequences such that \( \lim_{n \to \infty} d(p_n, q_n) = 0 \) and \( \lim_{n \to \infty} d(q_n, r_n) = 0 \). Let \( \varepsilon > 0 \) be
Let \( p, q \in X \). Then \( \triangle (P_p, P_q) = \lim_{n \to \infty} d(p_n, q_n) = d(p, q) \) by definition.

Let \( Y = \varphi(X) \) and let \( P = \{p_k\} \in X^* \) (where \( \{p_k\} \) is a representative from the equivalence class), supposing that \( P \notin Y \). Let \( N \) be a neighborhood of \( P \) with radius \( r \). There exists some \( M \) such that for all \( m, n \geq M \), \( d(p_m, p_n) < r \). Let \( Q = \varphi(p_M) \in Y \). We want to show that \( Q \in N \); we have \( d(p_n, p_M) < r \) whenever \( n \geq M \), and therefore

\[
\triangle(P, Q) = \lim_{n \to \infty} d(p_n, p_M) < r.
\]

This proves that \( \varphi(X) \) is dense in \( X^* \). **Second part incomplete.**
Chapter 4. Continuity

**Theorem 29.** Let $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ and let $a$ be a limit point of $X$. If $f(x) < g(x)$ for all $x$ in a neighborhood of $a$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x),$$

provided that both limits exist.

**Proof.** Let $N$ be a neighborhood of $a$ with radius $r$ such that $f(x) < g(x)$ for all $x \in N$. Suppose that $\lim_{x \to a}[g(x) - f(x)] = L < 0$. Then there exists a $\delta > 0$ such that $|g(x) - f(x) - L| < -L$ and $g(x) < f(x)$ whenever $0 < |x - a| < \delta$. Choose a point $x$ such that $0 < |x - a| < \min(\delta, r)$; this results in a contradiction. \hfill \Box

**Corollary 30.** Let $f, g : [a, \infty) \to \mathbb{R}$. If $f(x) \leq g(x)$ for all $x \geq a$, then

$$\lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x),$$

provided that both limits exist.

**Theorem 31.** [Theorem 4.8] A mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

**Proof.** Suppose that $f$ is continuous on $X$. Let $V$ be an open set in $Y$ and let $p \in f^{-1}(V)$. There exists a neighborhood $N$ of $f(p)$ with radius $r$ wholly contained in $V$. Since $f$ is continuous, there exists a $\delta > 0$ such that $d_Y(f(p), f(x)) < r$ whenever $x \in X$ and $d_X(p, x) < \delta$. Therefore, $N_\delta(p)$ is an open set of $X$ wholly contained in $f^{-1}(V)$. This shows that $f^{-1}(V)$ is an open set. Conversely, suppose that $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$. Let $p \in X$ and let $\varepsilon > 0$ be given. Let $V$ be a neighborhood of $f(p)$ with radius $\varepsilon$ so that $f^{-1}(V)$ is open in $X$. Since $p \in f^{-1}(V)$, there exists a neighborhood $N$ of $p$ with radius $\delta$ such that $N$ is wholly contained in $f^{-1}(V)$. Then for all $x \in X$ with $d_X(p, x) < \delta$, we have $d_Y(f(p), f(x)) < \varepsilon$ since $x \in f^{-1}(V)$ and $f(x) \in V$. This shows that $f$ is continuous on $X$. \hfill \Box

**Theorem 32.** [Examples 4.11] The map $x \mapsto |x|$ is continuous.

**Proof.** Let $\varepsilon > 0$ be given and let $x, y \in \mathbb{R}^k$ be arbitrary. Whenever $|x - y| < \varepsilon$, we have $||x|-|y|| \leq |x-y| < \varepsilon$, which completes the proof. \hfill \Box

**Theorem 33.** [Exercise 4.2] Let $f$ be a continuous map from a metric space $X$ to a metric space $Y$. Then for every set $E \subseteq X$,

$$f(E) \subseteq \overline{f(E)}.$$

Furthermore, this inclusion can be proper for certain functions.
Proof. Let \( p \in f(E) \); we must show that either \( p \in f(E) \) or \( p \) is a limit point of \( f(E) \). If there is a \( x \in E \) with \( p = f(x) \), then we are done. Otherwise, \( p \notin f(E) \), and we can choose \( x \) with \( p = f(x) \) such that \( x \) is a limit point of \( E \). Let \( N \) be a neighborhood of \( p \) with radius \( r \). Since \( f \) is continuous, there exists a \( \delta > 0 \) such that for all \( y \in N_\delta(x) \) we have \( f(y) \in N \). Since \( x \) is a limit point of \( E \), there exists a \( z \in N_\delta(x) \) with \( z \in E \) so that \( f(z) \in N \). Furthermore, \( f(z) \neq p \) since we assumed that \( p \notin f(E) \). This shows that \( p \) is a limit point of \( f(E) \).

The inclusion can be proper, as in the following example. Let \( f : (0,1) \to \mathbb{R} \) be defined by \( x \mapsto x \); then \( f \left( \left[ 0,1 \right] \right) = (0,1) \neq [0,1] = f \left( \left[ 0,1 \right] \right) \). \( \square \)

**Theorem 34.** [Exercise 4.3] Let \( f \) be a continuous map from a metric space \( X \) to \( \mathbb{R} \). Let \( Z(f) \) be the set of all \( p \in X \) such that \( f(p) = 0 \). Then \( Z(f) \) is closed.

**Proof.** By definition \( Z(f) = f^{-1} \{0\} \). Since \( \{0\} \) is closed and \( f \) is continuous, \( Z(f) \) must be closed. \( \square \)

**Theorem 35.** [Exercise 4.4] Let \( f \) and \( g \) be continuous mappings from a metric space \( X \) to a metric space \( Y \), and let \( E \) be a dense subset of \( X \). Then

1. \( f(E) \) is dense in \( f(X) \), and
2. If \( g(p) = f(p) \) for all \( p \in E \) then \( g(p) = f(p) \) for all \( p \in X \).

**Proof.** We know that \( \overline{E} \subseteq X \), and since \( E \) is dense in \( X \), \( X \subseteq \overline{E} \). By Theorem 33, we have \( f(E) = f(X) \subseteq \overline{f(E)} \), which shows that \( f(E) \) is dense in \( f(X) \).

To prove (2), let \( p \in X \). Since \( E \) is dense in \( X \), either \( p \in E \) or \( p \) is a limit point of \( E \). If \( p \in E \), then from the assumptions we are done. Otherwise, fix \( \varepsilon > 0 \). Since \( f \) is continuous, there exists a \( \delta_1 > 0 \) such that for every \( x \in N_{\delta_1}(p) \) we have \( f(x) \in N_\varepsilon(f(p)) \). Similarly, there exists a \( \delta_2 > 0 \) such that for every \( x \in N_{\delta_2}(p) \) we have \( g(x) \in N_\varepsilon(g(p)) \). Let \( \delta = \min(\delta_1, \delta_2) \). Since \( p \) is a limit point of \( E \), there exists a point \( z \in N_\delta(p) \) with \( z \in E \). Then \( f(z) \in N_\varepsilon(f(p)) \) and \( f(z) = g(z) \in N_\varepsilon(g(p)) \) so that

\[
d(f(p), g(p)) \leq d(f(p), f(z)) + d(f(z), g(p)) < 2\varepsilon.
\]

Since \( \varepsilon \) was arbitrary, \( f(p) = g(p) \). \( \square \)

**Theorem 36.** [Exercise 4.6] Let \( E \) be a subset of \( \mathbb{R} \). Define the graph of a function \( f : E \to \mathbb{R} \) to be the set \( \{(x, f(x)) \mid x \in E \} \). If \( E \) is compact, then a function \( f : E \to \mathbb{R} \) is continuous if and only if its graph is compact.
Proof. Let $G$ be the graph of $f$ and let $g : E \to G$ be given by $x \mapsto (x, f(x))$. Clearly, $g$ is a bijection by definition. Suppose that $f$ is continuous. Since $x \mapsto x$ is continuous, by Theorem 4.10 we have that $g$ is continuous. By Theorem 4.14, the image of $g$ is compact, which proves the result. Conversely, suppose that the graph $G$ is compact. Let $V$ be a closed set in $\mathbb{R}$; we want to show that $f^{-1}(V)$ is closed. Let $p$ be a limit point of $f^{-1}(V)$. By Theorem 3.2, there exists a sequence $\{p_n\}$ in $f^{-1}(V)$ that converges to $p$. Consider the sequence $\{(p_n, f(p_n))\}$; since $G$ is compact, some subsequence $\{(p_{n_i}, f(p_{n_i}))\}$ converges to some $(p, y) \in G$, and by definition, $y = f(p)$. Now $\{f(p_n)\}$ is a sequence in $V$, and since $V$ is closed and the sequence converges to $f(p)$, we have $f(p) \in V$. Therefore $p \in f^{-1}(V)$, which shows that $f^{-1}(V)$ is closed. □

Theorem 37. [Exercise 4.8] Let $E$ be a bounded set in $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Then $f$ is bounded on $E$. If $E$ is not bounded, then the conclusion does not necessarily hold.

Proof. We can choose $M, N$ so that $M < x < N$ for all $x \in E$. Since $f$ is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Choose $n$ so that $N - M + \delta > (n + 1)\delta \geq N - M$. For every $x \in E$, there is an integer $k$ with $0 \leq k \leq n$ such that $|M + k\delta - x| < \delta$. Then $|f(M + k\delta) - f(x)| < 1$ which means $|f(x)| < 1 + |f(M + k\delta)|$. Now take

$$P = \min_{0 \leq k \leq n} |f(M + k\delta)|$$

where $k = 0, 1, \ldots, n$; we have $|f(x)| < 1 + P$ for all $x \in E$ and hence $f$ is bounded on $E$.

To show that $E$ must be bounded for the conclusion to hold, choose $f(x) = x$, which is uniformly continuous, and $E = \mathbb{R}$. □

Theorem 38. [Exercise 4.9] Let $f : X \to Y$. Then the following statements are equivalent:

1. $f$ is uniformly continuous.
2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\text{diam } f(E) < \varepsilon$ whenever $E \subseteq X$ and $\text{diam } E < \delta$.

Proof. Obvious. □

Theorem 39. Let $X$ and $Y$ be metric spaces. Let $f : X \to Y$ be a continuous function. If $\{s_n\}$ is a sequence in $X$ that converges to $s$, then $\{f(s_n)\}$ converges to $f(s)$.

Proof. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that $d(f(s), f(x)) < \varepsilon$ whenever $d(s, x) < \delta$. Since $s_n \to s$, there exists a $N$ such that for all $n \geq N$ we
have \(d(s, s_n) < \delta\). Then \(d(f(s), f(s_n)) < \varepsilon\) whenever \(n \geq N\), which completes the proof.

**Theorem 40.** Let \(X, Y, Z\) be metric spaces. Let \(f : X \to Y\) be a function with \(\lim_{x \to a} f(x) = b\) and let \(g : Y \to Z\) be continuous at \(b\). Then \(\lim_{x \to a} g(f(x)) = g(b)\).

**Proof.** Let \(\varepsilon > 0\) be given. Choose \(\delta > 0\) such that \(d_Z(g(x), g(b)) < \varepsilon\) whenever \(d_Y(x, b) < \delta\), and choose \(\gamma > 0\) such that \(d_Y(f(x), b) < \delta\) whenever \(0 < d_X(x, a) < \gamma\). Then \(d_Z(g(f(x)), g(b)) < \varepsilon\) whenever \(0 < d_X(x, a) < \gamma\).

**Theorem 41.** Let \(X\) and \(Y\) be metric spaces. Let \(f : X \to Y\) be a function with

\[
\lim_{x \to a} f(x) = L.
\]

If \(F\) is any neighborhood of \(a\) and \(g : E \to F\) is a continuous bijection where \(g^{-1}(a)\) is a limit point of \(E\), then

\[
\lim_{x \to g^{-1}(a)} f(g(x)) = L.
\]

**Proof.** For every \(\varepsilon > 0\), there exists a \(\delta > 0\) such that \(d(f(x), L) < \varepsilon\) whenever \(0 < d(x, a) < \delta\). Since \(g\) is continuous on \(E\), there exists a \(\gamma > 0\) such that \(d(g(x), a) < \delta\) whenever \(d(x, g^{-1}(a)) < \gamma\). Then for all \(x\) with \(0 < d(x, g^{-1}(a)) < \gamma\) we have \(0 < d(g(x), a) < \delta\), noting that \(d(g(x), a) = 0\) if and only if \(d(x, g^{-1}(a)) = 0\), since \(g\) is a bijection. Therefore, \(d(f(g(x)), L) < \varepsilon\), which completes the proof.

**Theorem 42.** [Exercise 4.10] Let \(X\) be a compact metric space and let \(Y\) be a metric space. If \(f : X \to Y\) is a continuous function, then \(f\) is also uniformly continuous.

**Proof.** Suppose that \(f\) is not uniformly continuous. Then there exists a \(\varepsilon > 0\) such that for every \(\delta > 0\) we have some \(E \subseteq X\) with \(\text{diam } E < \delta\) such that \(\text{diam } f(E) \geq \varepsilon > \gamma\), where \(\gamma = \varepsilon/2\). Let \(\delta_n = 1/n\); for each \(n\) we have points \(p_n, q_n \in X\) such that \(d_X(p_n, q_n) < \delta_n\) and \(d_Y(f(p_n), f(q_n)) > \gamma\). Since \(X\) is compact, some subsequence \(\{p_n\}\) converges to a point \(p \in X\). By Theorem 39 the sequence \(\{f(p_n)\}\) converges to \(f(p)\). Similarly we have \(q_n \to p\) and \(f(q_n) \to f(p)\) upon application of Theorem 13 and Theorem 39. Now there exist integers \(M, N\) such that \(d_Y(f(p), f(p_n)) < \gamma/2\) whenever \(n_i \geq M\), and \(d_Y(f(p), f(q_n)) < \gamma/2\) whenever \(n_i \geq N\). Taking \(n_i\) to be an integer with \(n_i \geq \max(M, N)\), we find that

\[
d_Y(f(p_{n_i}), f(q_{n_i})) \leq d_Y(f(p_{n_i}), f(p)) + d_Y(f(p), f(q_{n_i})) < \gamma,
\]

which is a contradiction.

**Theorem 43.** [Exercise 4.11] Let \(X\) and \(Y\) be metric spaces. If \(f : X \to Y\) is a uniformly continuous function, then \(\{f(x_n)\}\) is a Cauchy sequence in \(Y\) for every Cauchy sequence \(\{x_n\}\) in \(X\).
Proof. Let \( \{x_n\} \) be a Cauchy sequence in \( X \). Let \( \varepsilon > 0 \) be given. Since \( f \) is uniformly continuous, there exists a \( \delta > 0 \) such that \( d(f(x), f(y)) < \varepsilon \) whenever \( d(x, y) < \delta \). Since \( \{x_n\} \) is a Cauchy sequence, there exists a \( N \) such that \( d(x_i, x_j) < \delta \) whenever \( i, j \geq N \). Then for all \( i, j \geq N \) we have \( d(f(x_i), f(x_j)) < \varepsilon \), which completes the proof. \( \square \)

**Theorem 44.** [Exercise 4.12] Let \( X, Y, Z \) be metric spaces. If \( f : X \to Y \) and \( g : Y \to Z \) are uniformly continuous functions, then \( h = g \circ f \) is uniformly continuous.

**Proof.** Let \( \varepsilon > 0 \) be given. There exists a \( \delta_1 > 0 \) such that \( d_Z(g(x), g(y)) < \varepsilon \) whenever \( d_Y(x, y) < \delta_1 \). There also exists a \( \delta_2 > 0 \) such that \( d_Y(f(x), f(y)) < \delta_1 \) whenever \( d_X(x, y) < \delta_2 \). Then for all \( x, y \) with \( d_X(x, y) < \delta_2 \) we have
\[
d_Y(f(x), f(y)) < \delta_1
\]
and
\[
d_Z(g(f(x)), g(f(y))) = d_Z(h(x), h(y)) < \varepsilon.
\]

**Lemma 45.** Let \( X, Y \) be metric spaces and let \( f : X \to Y \) be a uniformly continuous function. Let \( \{x_n\}, \{y_n\} \) be sequences in \( X \) that both converge to \( x \in X \). If \( f(x_n) \to y \) and \( f(y_n) \to z \), then \( y = z \).

**Proof.** Fix \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there is some \( \delta > 0 \) such that \( d(f(a), f(b)) < \varepsilon/3 \) whenever \( d(a, b) < \delta \). For some \( N \) we have \( d(x, x_n) < \delta/2 \) and \( d(x, y_n) < \delta/2 \) whenever \( n \geq N \), so that
\[
d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) < \delta
\]
and therefore \( d(f(x_n), f(y_n)) < \varepsilon/3 \) whenever \( n \geq N \). Furthermore, there exist integers \( N_1, N_2 \) such that \( d(y, f(x_n)) < \varepsilon/3 \) whenever \( n \geq N_1 \) and \( d(z, f(y_n)) < \varepsilon/3 \) whenever \( n \geq N_2 \). Setting \( n = \max \{N, N_1, N_2\} \), we have
\[
d(y, z) \leq d(y, f(x_n)) + d(f(x_n), z)
\]
\[
\leq d(y, f(x_n)) + d(f(x_n), f(y_n)) + d(f(y_n), z)
\]
\[
< \varepsilon.
\]
Since \( \varepsilon \) was arbitrary, \( y = z \). \( \square \)

**Theorem 46.** [Exercise 4.13] Let \( E \) be a dense subset of a metric space \( X \), and let \( f : E \to \mathbb{R} \) be a uniformly continuous function. Then \( f \) has a continuous extension from \( E \) to \( X \).
Lemma 48. If a function \( x \) exists a \( p \) \( f \) let

\[ \text{Proof.} \]

Theorem 47. \[ \text{Proof.} \]

Lemma 49. \[ \text{Proof.} \]

Theorem 48. \[ \text{Proof.} \]
for all $\delta > x$ to form a sequence

Suppose that $f(x) = \infty$.

Proof. Let $E$ be the set of all $x \in (a, b)$ such that $f(x-) < f(x+)$. For each $x \in E$, associate with $x$ a triple $(p, q, r)$:

1. Choose $p \in \mathbb{Q}$ so that $f(x-) < p < f(x+)$.
2. There exists a $\delta > 0$ such that $|f(t) - f(x-)| < p - f(x-)$ whenever $x - \delta < t < x$. Choose $q \in \mathbb{Q}$ so that $x - \delta < q < x$. Then whenever $a < q < t < x$ we have $f(t) < p$.
3. There exists a $\delta > 0$ such that $|f(x+) - f(t)| < f(x+) - p$ whenever $x < t < x + \delta$. Choose $r \in \mathbb{Q}$ so that $x < r < x + \delta$. Then whenever $x < t < r < b$ we have $f(t) > p$.

Now we must prove that each triple is associated with at most one $x \in E$. Let $x, y \in E$ such that $x, y$ are both associated with the triple $(p, q, r)$. We obtain four inequalities:

$f(t) < p$ whenever $a < q < t < x$,

$f(t) > p$ whenever $x < t < r < b$,

$f(t) < p$ whenever $a < q < t < y$,

$f(t) > p$ whenever $y < t < r < b$.

Suppose that $x < y$. We can choose $u$ with $x < u < y$. Since $x < u < r$, we have $f(u) > p$, and since $q < u < y$, we have $f(u) < p$, which is a contradiction. Similarly, we obtain a contradiction if $x > y$. Therefore $x = y$. Let $F$ be the set of all $x \in (a, b)$ such that $f(x-) > f(x+)$; we can again associate with $x \in F$ a triple $(p, q, r)$. For the last kind of simple discontinuity, let $G$ be the set of all $x \in (a, b)$ such that $f(x-) = f(x+)$ but $f(x) \neq f(x-), f(x+)$. For each $x \in G$, associate with $x$ a triple $(q, r)$ where $q, r$ are defined in a similar way to the triples $(p, q, r)$ associated with $E$. The sets $E, F, G$ are all countable, so the result follows.

Theorem 51. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the following property: if $f(a) < c < f(b)$, then $f(x) = c$ for some $x \in (a, b)$. Also, for every $r \in \mathbb{Q}$, the set of all $x$ with $f(x) = r$ is closed. Then $f$ is continuous.

Proof. Suppose that $f$ is not continuous. Then there exist $\varepsilon > 0$ and $x \in \mathbb{R}$ such that for all $\delta > 0$ we have $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$ for some $y$. Put $\delta_n = 1/n$ to form a sequence $x_n \to x$ while $|f(x) - f(x_n)| \geq \varepsilon$ for all $n$. Either $x_n$ has an infinite number of points with $f(x) < f(x_n)$, or an infinite number of points with $f(x_n) < f(x)$. Assume without loss of generality that the former holds, so that there

$N$ of $f(x)$ with radius $r$ such that $N \subseteq f(V)$. Then $f(x) + r/2 \in f(V)$, which means that $f(x') > M$ for some $x' \in V$. This is a contradiction, so $f$ must be monotonic. \hfill \square
exists a subsequence \( x_n \to x \) with \( f(x) + \varepsilon \leq f(x_n) \) for all \( n \). Let \( r \) be some rational number with \( f(x) < r < f(x) + \varepsilon \). For all \( n \) we have \( f(x) < r < f(x_n) \); by the given property of \( f \), there exists a \( t_n \in (x, x_n) \) with \( f(t_n) = r \), and with the sequence \( t_n \) converging to \( x \) since \( x_n \to x \). Let \( E \) be the set of all \( a \) with \( f(a) = r \). Since \( t_n \to x \) and \( f(t_n) = r \), we have that \( x \) is a limit point of \( E \). But \( f(x) < r \), so \( E \) is not closed. This is a contradiction, and therefore \( f \) must be continuous.

**Theorem 52.** [Exercise 4.20] If \( E \) is a nonempty subset of a metric space \( X \), define the distance from \( x \in X \) to \( E \) by

\[
p_E(x) = \inf_{z \in E} d(x, z).
\]

Then:

1. \( p_E(x) = 0 \) if and only if \( x \in \overline{E} \).
2. \( p_E \) is a uniformly continuous function on \( X \).

**Proof.** Suppose that \( p_E(x) = 0 \) and \( x \notin E \). Let \( N \) be a neighborhood of \( x \) with radius \( r \); by definition of the infimum, \( N \) contains a point \( z \in E \) with \( d(x, z) < r \) (and \( z \neq x \)). Hence \( x \) is a limit point of \( E \). Conversely, suppose that \( p_E(x) = L \) with \( L > 0 \). Clearly \( x \notin E \) since \( d(x, x) = 0 \). Also, \( x \) is not a limit point of \( E \) since the neighborhood \( N_L(x) \) contains no points in \( E \). Therefore \( x \notin \overline{E} \).

Fix \( x, y \in X \). Then for all \( z \in E \) we have

\[
p_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).
\]

Therefore \( d(y, z) \geq p_E(x) - d(x, y) \) for all \( z \), which means that \( p_E(y) \geq p_E(x) - d(x, y) \). Similarly, \( p_E(x) \geq p_E(y) - d(x, y) \), and thus

\[
|p_E(x) - p_E(y)| \leq d(x, y).
\]

Whenever \( d(x, y) < \varepsilon \) we have \( |p_E(x) - p_E(y)| < \varepsilon \), which shows that \( p_E \) is uniformly continuous. \( \square \)

**Theorem 53.** [Exercise 4.21] Let \( K \) and \( F \) be disjoint sets in a metric space \( X \), with \( K \) compact and \( F \) closed. Then there exists a \( \delta > 0 \) such that \( d(p, q) > \delta \) for all \( p \in K \) and \( q \in F \).

**Proof.** Consider the map \( p_F : K \to \mathbb{R} \) defined in Theorem 52. Suppose that \( p_F(x) = 0 \) for some \( x \in K \). Then by Theorem 52, \( x \in \overline{F} = F \), which is a contradiction. Therefore \( p_F(x) > 0 \) for all \( x \in K \). Let \( D = p_F(K) \); since \( K \) is compact, \( D \) is compact, and additionally \( D \) is closed by the Heine-Borel theorem. Since \( 0 \in D^c \) and \( D^c \) is open, there exists a neighborhood \( N \) of \( 0 \) with radius \( r > 0 \) such that \( N \subseteq D^c \). Therefore, \( p_F(x) \geq r \) for all \( x \in K \), and the result follows. \( \square \)
Theorem 54. [Exercise 4.23] If \( f : (a, b) \to \mathbb{R} \) is a convex function and \( a < s < t < u < b \), then

\[
\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}
\]

and \( f \) is continuous. Additionally, every increasing convex function of a convex function is convex.

Proof. We have

\[
t = \frac{t - s}{u - s}u + \left(1 - \frac{t - s}{u - s}\right)s = \frac{u - t}{u - s}s + \left(1 - \frac{u - t}{u - s}\right)u.
\]

Then

\[
f(t) \leq \frac{t - s}{u - s}f(u) + \left(1 - \frac{t - s}{u - s}\right)f(s)
\]

and

\[
\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}
\]

and

\[
f(t) \leq \frac{u - t}{u - s}f(s) + \left(1 - \frac{u - t}{u - s}\right)f(u)
\]

\[
\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.
\]

Let \( x \in (a, b) \) and choose \( \delta \) so that \([x - \delta, x + \delta] \in (a, b)\). Let \( y \in (x - \delta, x + \delta) \setminus \{x\}\). We want to show that the following inequality holds:

\[
\frac{f(x) - f(x - \delta)}{\delta} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(x + \delta) - f(x)}{\delta}.
\]

If \( y < x \), then applying \([\ast]\) on \( x - \delta < y < x \) and \( y < x < x + \delta \) produces the result. Similarly, if \( y > x \) then applying \([\ast]\) on \( x - \delta < x < y \) and \( x < y < x + \delta \) produces the result. Then for all \( y \in (x - \delta, x + \delta) \), \( |f(x) - f(y)| \leq C|x - y| \) for some positive constant \( C \). This proves that \( f \) is continuous.

Let \( g : (c, d) \to \mathbb{R} \) be an increasing convex function where the range of \( f \) is a subset of \((c, d)\). Then for all \( x, y \in (a, b) \) and \( \lambda \in (0, 1) \),

\[
\frac{f(\lambda x + (1 - \lambda) y)}{\delta} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(x + \delta) - f(x)}{\delta},
\]

\[
\frac{g(f(\lambda x + (1 - \lambda) y))}{\lambda g(f(x)) + (1 - \lambda) g(f(y))}
\]

\[
\leq \lambda g(f(x)) + (1 - \lambda) g(f(y))\).
\]
which shows that $g \circ f$ is convex.

□

**Definition 55.** Let $I$ be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is **midpoint convex** if

$$f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in I$. A **binary sequence** is a sequence $\{b_n\}$ where every $b_n$ is either 0 or 1.

**Lemma 56.** Let $f : [0, 1] \to \mathbb{R}$ be a midpoint convex function and let $\{b_n\}$ be a binary sequence. Let $\lambda_n = \sum_{k=1}^{n} b_k 2^{-k}$. Then

$$f(\lambda_n) \leq \lambda_n f(1) + (1 - \lambda_n) f(0).$$

**Proof.** We first use induction on $n$ to prove that

$$f \left( \sum_{k=1}^{n} b_k 2^{-k} \right) \leq \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0)2^{-n}$$

for any binary sequence $\{b_n\}$. If $n = 1$ and $b_1 \in \{0, 1\}$, then

$$f \left( \frac{b_1}{2} \right) = f \left( \frac{0 + b_1}{2} \right) \leq \frac{1}{2} f (b_1) + \frac{1}{2} f(0)$$

since $f$ is midpoint convex. Otherwise, assuming the statement for $n - 1$, we have for any binary sequence $\{b_n\}$,

$$f \left( \sum_{k=1}^{n} b_k 2^{-k} \right) \leq \frac{1}{2} f(b_1) + \frac{1}{2} f \left( \sum_{k=2}^{n} b_k 2^{-k+1} \right)$$

$$\leq \frac{1}{2} f(b_1) + \frac{1}{2} f \left( \sum_{k=2}^{n} b_k 2^{-k+1} \right)$$

$$\leq \frac{1}{2} f(b_1) + \frac{1}{2} \sum_{k=2}^{n} f(b_k) 2^{-k+1} + f(0)2^{-n}$$

$$= \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0)2^{-n},$$
which proves the statement for all $n$. We now compute

$$1 - \lambda_n = \sum_{k=1}^{\infty} 2^{-k} - \sum_{k=1}^{n} b_k 2^{-k}$$

$$= \sum_{k=1}^{n} (1 - b_k) 2^{-k} + \sum_{k=n+1}^{\infty} 2^{-k}$$

$$= \sum_{k=1}^{n} (1 - b_k) 2^{-k} + 2^{-n}$$

so that

$$\lambda_n f(1) + (1 - \lambda_n) f(0) = \sum_{k=1}^{n} f(1)b_k 2^{-k} + \sum_{k=1}^{n} f(0)(1 - b_k) 2^{-k} + f(0)2^{-n}$$

$$= \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0)2^{-n}$$

since $b_k$ is always 0 or 1, and $f(1)b_k + f(0)(1 - b_k)$ is always equal to $f(b_k)$. This proves the result. \(\square\)

**Theorem 57.** [Exercise 4.24] Let $f : (a, b) \to \mathbb{R}$ be a continuous, midpoint convex function. Then $f$ is convex.

*Proof.* We first prove a smaller result for any continuous, midpoint convex function $g : [0, 1] \to \mathbb{R}$. Let $\lambda \in (0, 1)$ and let $\{b_n\}$ be a binary expansion of $\lambda$ so that if $\lambda_n = \sum_{k=1}^{n} b_k 2^{-k}$, then $\lambda_n \to \lambda$. By Lemma 56, we have $g(\lambda_n) \leq \lambda_n g(1) + (1 - \lambda_n)g(0)$, and by Theorem 39, $g(\lambda_n) \to g(\lambda)$. Therefore by Theorem 10

$$g(\lambda) \leq \lambda g(1) + (1 - \lambda)g(0).$$

(\*)

For the general case, let $x, y \in (a, b)$ and let $\lambda \in (0, 1)$. If $x = y$, then we are done. Otherwise, assume without loss of generality that $x < y$. Define $g : [0, 1] \to \mathbb{R}$ by $g(\lambda) = f(\lambda y + (1 - \lambda)x)$. For any $\lambda_1, \lambda_2 \in [0, 1]$, we have

$$g\left(\frac{\lambda_1 + \lambda_2}{2}\right) = f\left(x + \frac{\lambda_1 + \lambda_2}{2}(y-x)\right)$$

$$= f\left(\frac{[\lambda_1 y + (1 - \lambda_2)x] + [\lambda_2 y + (1 - \lambda_2)x]}{2}\right)$$

$$\leq \frac{g(\lambda_1) + g(\lambda_2)}{2},$$
which shows that $g$ is midpoint convex. By [1],
\[
g(\lambda) \leq \lambda g(1) + (1 - \lambda)g(0)
f(\lambda y + (1 - \lambda)x) \leq \lambda f(y) + (1 - \lambda)f(x)
\]
for all $\lambda \in (0, 1)$. This proves that $f$ is convex. □

**Theorem 58.** [Exercise 4.26] Let $X, Y, Z$ be metric spaces with $Y$ compact. Let $f : X \to Y$ such that $f(X) \subseteq Y$, and let $g : Y \to Z$ be a continuous, injective function. Let $h : X \to Z$ be defined by $h(x) = g(f(x))$. Then:

1. If $h$ is uniformly continuous, then $f$ is uniformly continuous.
2. If $h$ is continuous, then $f$ is continuous.

**Proof.** Suppose that $h$ is uniformly continuous. Since $g$ is continuous and $Y$ is compact, $g(Y)$ is compact. Since $g$ is injective, $f(x) = g^{-1}(h(x))$, and $g^{-1} : g(Y) \to Y$ is continuous by Theorem 4.17. But $g(Y)$ is compact, so by Theorem 4.19, $g^{-1}$ is uniformly continuous. Applying Theorem 4.14 proves that $f$ is uniformly continuous.

Suppose that $h$ is continuous. Again, $f = g^{-1} \circ h$, and $g^{-1}$ is continuous by Theorem 4.17. Applying Theorem 4.7 proves that $f$ is continuous. □

**Chapter 5. Differentiation**

**Lemma 59.** Let $I$ be an interval and let $f : I \to \mathbb{R}$ be a function differentiable at $x$. Then there exists a function $\phi : I \to \mathbb{R}$ such that
\[
f(t) - f(x) = (t - x)[f'(x) + \phi(t)]
\]
for all $t \in I$ and
\[
\lim_{t \to x} \phi(t) = \phi(0) = 0.
\]

**Proof.** Define
\[
\phi(t) = \begin{cases} 
0 & \text{if } t = x, \\
\frac{f(t) - f(x)}{t - x} - f'(x) & \text{otherwise.}
\end{cases}
\]
This function clearly satisfies the desired properties. □

**Theorem 60.** Let $I_1, I_2$ be intervals. Let $f : I_1 \to \mathbb{R}$ be a continuous function and let $g : I_2 \to \mathbb{R}$ be a function where $I_2$ contains the range of $f$. Define $h : I_1 \to \mathbb{R}$ by $h(x) = g(f(x))$. If $f$ is differentiable at some point $x \in I_1$ and $g$ is differentiable at $f(x)$, then $h'(x) = g'(f(x))f'(x)$. 
Proof. Let y = f(x) for convenience. By Lemma 59 there exist functions φ₁, φ₂ with

\[ \lim_{t \to x} \phi_1(t) = \lim_{s \to y} \phi_2(s) = 0 \]

such that

\[
\begin{align*}
f(t) - f(x) &= (t - x)[f'(x) + \phi_1(t)], \\
g(s) - g(y) &= (s - y)[g'(y) + \phi_2(s)],
\end{align*}
\]

whenever \( t \in I_1 \) and \( s \in I_2 \). In particular, by setting \( s = f(t) \) we have for all \( t \in I_1 \),

\[
h(t) - h(x) = g(f(t)) - g(f(x))
= (f(t) - f(x))[g'(f(x)) + \phi_2(f(t))]
= (t - x)[f'(x) + \phi_1(t)][g'(f(x)) + \phi_2(f(t))],
\]

so that

\[
(*) \quad \frac{h(t) - h(x)}{t - x} = [f'(x) + \phi_1(t)][g'(f(x)) + \phi_2(f(t))]
\]

if \( t \neq x \). By Theorem 40

\[ \lim_{t \to x} \phi_2(f(t)) = \phi_2(f(x)) = 0 \]

since \( f \) is continuous at \( x \) and \( \phi_2 \) is continuous at \( f(x) \), so taking \( t \to x \) in (*) completes the proof. \( \square \)

**Theorem 61.** [Exercise 5.1] Let \( f \) be defined for all real \( x \), and suppose that

\[ |f(x) - f(y)| \leq (x - y)^2 \]

for all real \( x \) and \( y \). Then \( f \) is constant.

**Proof.** The condition on \( f \) is that

\[ \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y| \]

for all \( x, y \in \mathbb{R} \). Then \( f'(x) = 0 \) for all \( x \), and by the mean value theorem, \( f \) is constant. \( \square \)

**Theorem 62.** [Exercise 5.2] Let \( f : (a, b) \to \mathbb{R} \) with \( f'(x) > 0 \) for all \( x \in (a, b) \). Then:

1. \( f \) is strictly increasing in \( (a, b) \), and
2. If \( g \) is the inverse function of \( f \), then \( g \) is differentiable and

\[ g'(f(x)) = \frac{1}{f'(x)} \]

for all \( x \in (a, b) \).
Proof. Let \( x, y \in (a, b) \) with \( x < y \). By the mean value theorem, there exists a \( c \in (x, y) \) such that \( f(y) - f(x) = (y - x)f'(c) > 0 \), and therefore \( f(x) < f(y) \). This shows that \( f \) is strictly increasing in \((a, b)\). Let \( x \in (a, b) \); we want to show that \( g \) is differentiable at \( f(x) \). Since \( f \) is differentiable at \( x \), we have
\[
\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x).
\]
By Theorem 4.4, since \( f'(x) > 0 \),
\[
\lim_{t \to x} \frac{t - x}{f(t) - f(x)} = \frac{1}{f'(x)}.
\]
By Theorem 4.4 applied with \( g \), we have
\[
\lim_{t \to f(x)} \frac{g(t) - g(f(x))}{t - f(x)} = \frac{1}{f'(x)}
\]
and therefore \( g'(f(x)) = 1/f'(x) \).

**Theorem 63.** [Exercise 5.3] Let \( g : \mathbb{R} \to \mathbb{R} \) with a bounded derivative \( |g'| \leq M \). Fix \( \varepsilon > 0 \) and let \( f(x) = x + \varepsilon g(x) \). Then \( f \) is injective if \( \varepsilon \) is small enough.

Proof. Take \( \varepsilon < 1/M \). Let \( x, y \in \mathbb{R} \) such that \( f(x) = f(y) \), i.e. \( x + \varepsilon g(x) = y + \varepsilon g(y) \), so that
\[
\left| \frac{g(x) - g(y)}{x - y} \right| = \frac{1}{\varepsilon}.
\]
Suppose that \( x \neq y \); then by the mean value theorem, there exists a \( z \in (x, y) \) such that
\[
|g'(z)| = \left| \frac{g(x) - g(y)}{x - y} \right| = \frac{1}{\varepsilon} \leq M.
\]
This is a contradiction since \( 1/\varepsilon > M \), so \( x = y \) whenever \( f(x) = f(y) \).

**Theorem 64.** [Exercise 5.4] If \( C_0, \ldots, C_n \) are real constants such that
\[
C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,
\]
then the equation
\[
C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0
\]
has at least one real root between 0 and 1.

Proof. Let
\[
f(x) = C_0x + \frac{C_1}{2}x + \cdots + \frac{C_{n-1}}{n}x^{n-1} + \frac{C_n}{n+1}x^{n+1}
\]
so that \( f(0) = f(1) = 0 \). By the mean value theorem, there exists a \( x \in (0, 1) \) such that
\[
f'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0.
\]

\[\square\]

**Theorem 65.** [Exercise 5.5] Let \( f \) be defined and differentiable for every \( x > 0 \), with \( f'(x) \to 0 \) as \( x \to +\infty \). Let \( g(x) = f(x + 1) - f(x) \). Then \( g(x) \to 0 \) as \( x \to +\infty \).

**Proof.** For every \( \varepsilon > 0 \), there exists a \( M > 0 \) such that \( |f'(x)| < \varepsilon \) whenever \( x > M \). Then for all \( x > M \), applying the mean value theorem to \( f \) gives a \( c \in (x, x + 1) \) such that \( f(x + 1) - f(x) = f'(c) \). Since \( c > M \), we have \( |f(x + 1) - f(x)| = |f'(c)| < \varepsilon \), which proves that \( g(x) \to 0 \) as \( x \to +\infty \). \[\square\]

**Theorem 66.** [Exercise 5.6] Let \( f \) be a real function. Suppose that

1. \( f \) is continuous for \( x \geq 0 \),
2. \( f'(x) \) exists for \( x > 0 \),
3. \( f(0) = 0 \),
4. \( f' \) is monotonically increasing.

Let \( g(x) = \frac{f(x)}{x} \) be defined for all \( x > 0 \). Then \( g \) is monotonically increasing.

**Proof.** The derivative of \( g \) is given by
\[
g'(x) = \frac{x f'(x) - f(x)}{x^2},
\]
so we want to prove that \( x f'(x) - f(x) > 0 \) for all \( x > 0 \). For all \( x > 0 \), by the mean value theorem, there exists a \( c \in (0, x) \) such that
\[
\frac{f(x)}{x} = f'(c) < f'(x)
\]
since \( c < x \) and \( f' \) is monotonically increasing. This proves the result. \[\square\]

**Theorem 67.** [Exercise 5.7] Suppose that \( f'(x) \) and \( g'(x) \) exist, \( g'(x) \neq 0 \), and \( f(x) = g(x) = 0 \). Then
\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.
\]


Proof. Since \( f'(x) \) and \( g'(x) \) exist, we have

\[
\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{f(t)}{t - x} = f'(x),
\]

\[
\lim_{t \to x} \frac{g(t) - g(x)}{t - x} = \lim_{t \to x} \frac{g(t)}{t - x} = g'(x).
\]

Since \( g'(x) \neq 0 \), by Theorem 4.4 the result follows. \( \square \)

**Theorem 68.** [Exercise 5.8] Suppose that \( f' \) is continuous on \([a,b]\) and \( \varepsilon > 0 \). Then there exists a \( \delta > 0 \) such that

\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon
\]

whenever \( 0 < |t - x| < \delta \) and \( t, x \in [a,b] \).

Proof. By Theorem 4.19, \( f' \) is uniformly continuous since \([a,b]\) is compact. There exists a \( \delta > 0 \) such that \( |f'(t) - f'(x)| < \varepsilon \) whenever \( |t - x| < \delta \). Then for all \( t, x \in [a,b] \) with \( 0 < |t - x| < \delta \), by the mean value theorem, there exists a \( u \in (t, x) \) such that

\[
\left| \frac{f(t) - f(x)}{t - x} - f'(u) \right| = 0,
\]

and

\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| \leq \left| \frac{f(t) - f(x)}{t - x} - f'(u) \right| + |f'(u) - f'(c)|
\]

\[
< \varepsilon.
\]

\( \square \)

**Theorem 69.** [Exercise 5.9] Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( f'(x) \) exists for all \( x \neq 0 \) and \( f'(x) \to 3 \) as \( x \to 0 \). Then \( f'(0) \) exists.

Proof. For every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f'(x) - 3| < \varepsilon \) whenever \( 0 < |x| < \delta \). For all \( x \) with \( 0 < |x| < \delta \), by the mean value theorem, there exists a \( c \in (0, x) \) such that

\[
\frac{f(x) - f(0)}{x} = f'(c)
\]

\[
\left| \frac{f(x) - f(0)}{x} - 3 \right| = |f'(c) - 3| < \varepsilon.
\]
Theorem 70. [Exercise 5.11] Suppose that \( f \) is defined in a neighborhood of \( x \), and suppose that \( f''(x) \) exists. Then
\[ \lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x). \]

Proof. Since \( f''(x) \) exists, we have
\[ f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h} = \lim_{h \to 0} \frac{f'(x) - f'(x - h)}{h} \]
where the second limit is obtained by applying Theorem 41 with the bijection \( h \mapsto -h \). Adding the two limits gives
\[ f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h}. \]
As \( h \to 0 \) we have \( f(x + h) + f(x - h) - 2f(x) \to 0 \) and \( h^2 \to 0 \), so by Theorem 5.13,
\[ \lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h} = f''(x). \]

Theorem 71. [Exercise 5.14] Let \( f : (a,b) \to \mathbb{R} \) be a differentiable function. Then \( f \) is convex if and only if \( f' \) is monotonically increasing. If \( f''(x) \) exists for all \( x \in (a,b) \), then \( f \) is convex if and only if \( f''(x) \geq 0 \) for all \( x \in (a,b) \).

Proof. Suppose that \( f \) is convex. Let \( x, y \in (a,b) \) with \( x < y \). Since \( f \) is convex, every \( t \in (x, y) \) has
\[ \frac{f(t) - f(x)}{t - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(t)}{y - t}. \]
Then
\[ \lim_{t \to x^+} \frac{f(t) - f(x)}{t - x} \leq \lim_{t \to y^-} \frac{f(y) - f(t)}{y - t}, \]
and since \( f'(x), f'(y) \) both exist, \( f'(x) \leq f'(y) \). Conversely, suppose that \( f' \) is monotonically increasing. Let \( x, y \in (a,b) \) with \( x < y \) and let \( \lambda \in (0, 1) \). Let \( t = (1 - \lambda)x + \lambda y \).
By the mean value theorem,
\[
\frac{f(t) - f(x)}{t - x} = f'(t_1) \\
\frac{f(y) - f(t)}{y - t} = f'(t_2)
\]
for some \(t_1 \in (x, t)\) and \(t_2 \in (t, y)\). Since \(t_1 < t_2\),
\[
\frac{f(t) - f(x)}{t - x} \leq \frac{f(y) - f(t)}{y - t}
\]
\[
(1 - \lambda)(y - x)(f(t) - f(x)) \leq \lambda(y - x)(f(y) - f(t))
\]
\[
f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y),
\]
which shows that \(f\) is convex. If \(f''\) is defined on \((a, b)\), then \(f'\) is monotonically increasing if and only if \(f''(x) \geq 0\) for all \(x \in (a, b)\).

\[\square\]

**Theorem 72.** [Exercise 5.15] Let \(a \in \mathbb{R}\) and suppose that \(f : (a, \infty) \to \mathbb{R}\) is twice-differentiable. Suppose that \(M_0, M_1, M_2\) are the least upper bounds of \(|f(x)|, |f'(x)|, |f''(x)|\) respectively on \((a, \infty)\). Then \(M_1^2 \leq 4M_0M_2\).

**Proof.** Let \(x \in (a, \infty)\). For any \(h > 0\), by Theorem 5.15, there exists a point \(\xi \in (x, x + 2h)\) such that
\[
f(x + 2h) = f(x) + 2hf'(x) + 2h^2f''(\xi)
\]
\[
f'(x) = \frac{1}{2h} [f(x + 2h) - f(x)] - hf''(\xi).
\]
Then
\[
|f'(x)| \leq \left| \frac{1}{2h} [f(x + 2h) - f(x)] - hf''(\xi) \right|
\]
\[
\leq \frac{|f(x + 2h)| + |f(x)|}{2h} + h|f''(\xi)|
\]
\[
\leq hM_2 + \frac{M_0}{h}
\]
so that \(M_1 \leq hM_2 + M_0/h\) since \(M_1\) is the least upper bound of \(|f'(x)|\). Setting \(h = M_1/(2M_2)\) gives \(M_1^2 \leq 4M_0M_2\). \[\square\]

**Theorem 73.** [Exercise 5.16] Suppose that \(f : (0, \infty) \to \mathbb{R}\) is twice-differentiable, \(f''\) is bounded on \((0, \infty)\), and \(f(x) \to 0\) as \(x \to \infty\). Then \(f'(x) \to 0\) as \(x \to \infty\).

**Proof.** Choose \(M\) such that \(|f''(x)| \leq M\) for all \(x \in (0, \infty)\). Let \(\varepsilon > 0\) be given. There exists a \(A\) such that \(|f(x)| < \varepsilon^2/(16M)\) for all \(x \in (A, \infty)\), and by Theorem 72 we have \(|f'(x)| \leq \varepsilon/2 < \varepsilon\) for all \(x \in (A, \infty)\). This shows that \(f'(x) \to 0\) as \(x \to \infty\). \[\square\]
Theorem 74. [Exercise 5.17] Suppose that \( f : [-1,1] \to \mathbb{R} \) is a three times differentiable function such that
\[
f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.
\]
Then \( f^{(3)}(x) \geq 3 \) for some \( x \in (-1,1) \).

Proof. By Theorem 5.15, there exist points \( s \in (0,1) \) and \( t \in (-1,0) \) such that
\[
f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6},
\]
(*)
\[
1 = \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6},
\]
\[
f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6},
\]
(**)
\[
0 = \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6}.
\]
Subtracting (**) from (*) gives \( f^{(3)}(s) + f^{(3)}(t) = 6 \). If \( f^{(3)}(s) \geq 3 \) then we are done; otherwise, \( f^{(3)}(s) = 6 - f^{(3)}(t) < 3 \), so \( f^{(3)}(t) > 3 \). \( \square \)

Theorem 75. [Exercise 5.18] Let \( n \) be a positive integer. Suppose that for \( f : [a,b] \to \mathbb{R} \), the value \( f^{(n-1)}(t) \) exists for every \( t \in [a,b] \). Let \( \alpha, \beta \), and \( P \) be as in Theorem 5.15. Define \( Q(t) = (f(t) - f(\beta)) / (t - \beta) \) for all \( t \in [a,b] \) and \( t \neq \beta \). Then
\[
f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n.
\]

Proof. We want to prove that
\[
\frac{Q^{(n-1)}(t)}{(n-1)!} (\beta - t)^n = f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!} (\beta - t)^k
\]
for all \( n \geq 1 \). The case \( n = 1 \) is equivalent to the definition of \( Q \). Assuming the statement for \( n \) and differentiating the above expression, we have
\[
\frac{Q^{(n)}(t)}{(n-1)!} (\beta - t)^n - \frac{Q^{(n-1)}(t)}{(n-1)!} \cdot n(\beta - t)^{n-1} = -\frac{f^{(n)}(t)}{(n-1)!} (\beta - t)^{n-1}
\]
\[
Q^{(n)}(t) (\beta - t)^{n+1} = \frac{Q^{(n-1)}(t)}{(n-1)!} \cdot n(\beta - t)^n - \frac{f^{(n)}(t)}{(n-1)!} (\beta - t)^n,
\]
\[
Q^{(n)}(t) (\beta - t)^{n+1} = \frac{Q^{(n-1)}(t)}{(n-1)!} \cdot n(\beta - t)^n - \frac{f^{(n)}(t)}{(n-1)!} (\beta - t)^n,
\]
Proof. Suppose that $f$ has no fixed point, but $f′(t) \neq 1$ for all $t \in \mathbb{R}$. Then $f$ has at most one fixed point.

**Theorem 76.** [Exercise 5.22(a)] Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f′(t) \neq 1$ for all $t \in \mathbb{R}$. Then $f$ has at most one fixed point.

**Proof.** Suppose that $f$ has two fixed points, $x = f(x)$ and $y = f(y)$, with $x \neq y$. By the mean value theorem, there exists a $c \in (x, y)$ such that
\[
\frac{f(y) - f(x)}{y - x} = 1 = f′(c),
\]
which is a contradiction. \hfill \square

**Theorem 77.** [Exercise 5.22(b)] Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) = t + (1 + e^t)^{-1}$. Then $f$ has no fixed point, but $f′(t) \in (0, 1)$ for all $t \in \mathbb{R}$.

**Proof.** To show that $f$ has no fixed point, note that $(1 + e^t)^{-1} \neq 0$ for all $t \in \mathbb{R}$, so that $f(t) = t + (1 + e^t)^{-1} \neq t$ for all $t \in \mathbb{R}$. Also,
\[
f′(t) = 1 - \frac{e^t}{(1 + e^t)^2}
\]
\[
= 1 - \frac{1}{1 + e^t} + \frac{1}{(1 + e^t)^2}.
\]
From the first line, $f′(t) < 1$ for all $t \in \mathbb{R}$, and from the second line, $f′(t) > 0$ for all $t \in \mathbb{R}$. \hfill \square

**Theorem 78.** [Exercise 5.22(c)] Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If there exists a constant $A < 1$ such that $|f′(t)| \leq A$ for all $t \in \mathbb{R}$, then $f$ has a fixed point $x = \lim_{n \to \infty} x_n$ where $x_0 \in \mathbb{R}$ is arbitrary and $x_{n+1} = f(x_n)$ for $n \geq 0$.

**Proof.** The case $A = 0$ is trivial, so we may assume that $A > 0$. By the mean value theorem, $|f(x) - f(y)| \leq A|x - y|$ for all $x, y \in \mathbb{R}$. In particular, $|x_{i+1} - x_{j+1}| \leq A|x_i - x_j|$ for all $i, j \geq 0$, and $|x_m - x_{m-1}| \leq A^{m-1}|x_1 - x_0|$ for all $m \geq 1$. We now prove that for all $n \geq 1$,
\[
|x_{m+n} - x_m| \leq A(1 - A^n)\frac{1}{1 - A}|x_m - x_{m-1}|.
\]
The case \( n = 1 \) is clear. Assuming the statement for \( n - 1 \), we have

\[
|x_{m+n} - x_m| \leq |x_{m+n-1} - x_m| + |x_{m+n} - x_{m+n-1}|
\]

\[
\leq \frac{A(1 - A^{n-1})}{1 - A} |x_m - x_{m-1}| + A^n |x_m - x_{m-1}|
\]

\[
= \frac{A(1 - A^n)}{1 - A} |x_m - x_{m-1}|,
\]

which proves the statement for all \( n \geq 1 \). Furthermore,

\[
|x_{m+n} - x_m| < \frac{A}{1 - A} |x_m - x_{m-1}|
\]

for all \( n \geq 1 \). Let \( \varepsilon > 0 \) be given. Recall that \( |x_m - x_{m-1}| \leq A^{m-1} |x_1 - x_0| \) for all \( m \geq 1 \) and that \( A < 1 \); there exists a \( N \) such that \( |x_k - x_{k-1}| \leq \varepsilon (1 - A)/A \) for all \( k \geq N \). Let \( m, n \geq N \) and assume without loss of generality that \( m < n \). Then

\[
|x_n - x_m| = |x_{m+(n-m)} - x_m|
\]

\[
< \frac{A}{1 - A} |x_m - x_{m-1}|
\]

\[
< \varepsilon,
\]

which shows that \( \{x_n\} \) is a Cauchy sequence. By Theorem 3.11, \( \{x_n\} \) converges to some value \( x \); we want to show that \( x \) is indeed a fixed point of \( f \). Fix \( \varepsilon > 0 \). We know that \( x_n \to x \), \( \{x_n\} \) is a Cauchy sequence, and \( f(x_n) \to f(x) \) because \( f \) is continuous. Then there exists some integer \( n \) such that

\[
|x - f(x)| \leq |x - x_n| + |x_n - f(x_n)| + |f(x_n) - f(x)|
\]

\[
= |x - x_n| + |x_n - x_{n+1}| + |f(x_n) - f(x)|
\]

\[
< 3\varepsilon.
\]

Since \( \varepsilon \) was arbitrary, \( x = f(x) \).  

\[ \square \]

**Theorem 79.** [Exercise 5.23] The function \( f(x) = (x^3 + 1)/3 \) has three fixed points \( \alpha, \beta, \gamma \), where \( -2 < \alpha < -1, 0 < \beta < 1, \) and \( 1 < \gamma < 2 \). For an arbitrarily chosen \( x_1 \), define \( \{x_n\} \) by setting \( x_{n+1} = f(x_n) \).

1. If \( x_1 < \alpha \), then \( x_n \to -\infty \) as \( n \to \infty \).
2. If \( \alpha < x_1 < \gamma \), then \( x_n \to \beta \) as \( n \to \infty \).
3. If \( \gamma < x_1 \), then \( x_n \to +\infty \) as \( n \to \infty \).
Proof. Let \( g(x) = x^3 - 3x + 1 \); since \( \alpha, \beta, \gamma \) are fixed points of \( f \), they are roots of \( g \). Suppose that \( x_1 < \alpha \). For any \( c > 0 \), we can compute

\[
g(\alpha - c) = (\alpha^3 - 3\alpha + 1) - 3\alpha^2 c + 3\alpha c^2 - c^3 - 3c\]

\[
= c(3(1 - \alpha^2) + 3\alpha c - c^2) < 3\alpha c^2 - c^3 < -c^3
\]

(*) \( f(\alpha - c) < (\alpha - c) - \frac{c^3}{3} \).

Let \( d = \alpha - x_1 > 0 \); (*) shows that \( x_{n+1} < x_n - d/3 \) for every \( n \geq 1 \), and clearly \( x_n \to -\infty \) as \( n \to \infty \). Now suppose that \( \alpha < x_1 < \gamma \). A simple induction argument shows that \( \alpha < x_n < \gamma \) for all \( n \geq 1 \), and by a variation on Theorem 78, \( x_n \to \beta \) since \( f'(x) = x^2 \in [0, \max(\alpha, \gamma)] \) for all \( x \in [\alpha, \gamma] \). Finally, the case for \( \gamma < x_1 \) is similar to the case \( x_1 < \alpha \). □

Proposition 80. [Exercise 5.25] Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function with \( f(a) < 0, f(b) > 0, f'(x) \geq \delta > 0 \), and \( 0 < f''(x) \leq M \) for all \( x \in [a, b] \). Let \( \xi \) be the unique point in \( (a, b) \) at which \( f(\xi) = 0 \). [Note: the inequality \( 0 \leq f''(x) \) has been changed to \( 0 < f''(x) \).]

Choose \( x_1 \in (\xi, b) \) and define \( \{x_n\} \) by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

We now prove by induction that \( x_{n+1} \in (\xi, x_n) \) for all \( n \). For all \( n \), applying the mean value theorem gives a value \( c \in (\xi, x_n) \) such that

\[
\frac{f(x_n) - f(\xi)}{x_n - \xi} = f'(c) < f'(x_n),
\]

since \( c < x_n \) and \( f' \) is strictly increasing. Therefore

\[
f(x_n) < f(x_n) - f'(x_n)(x_n - x_{n+1}) < x_n - x_{n+1} < x_n - \xi
\]

and \( \xi < x_{n+1} \). Also, \( f(x_n) > 0 \) for otherwise \( f(y) = 0 \) for some \( y \in [x_n, b) \) by the intermediate value theorem. Therefore \( f(x_n)/f'(x_n) > 0 \) and \( x_{n+1} < x_n \).
Applying Taylor’s theorem with \( \alpha = x_n \), \( \beta = \xi \) gives a point \( t_n \in (\xi, x_n) \) such that

\[
f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{1}{2}f''(t_n)(\xi - x_n)^2
\]

\[
x_n - \frac{f(x_n)}{f'(x_n)} = \xi + \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\]

\[
x_n+1 - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2.
\]

Consider the statement

\[
x_{n+1} - \xi \leq \frac{1}{A} \left[A(x_1 - \xi)\right]^{2^n}.
\]

If \( n = 1 \), then

\[
x_2 - \xi = \frac{f''(t_n)}{2f'(x_1)}(x_1 - \xi)^2
\]

\[
\leq \frac{M}{2f'(x_1)}(x_1 - \xi)^2
\]

\[
< A(x_1 - \xi)^2.
\]

Otherwise, assuming the statement for \( n - 1 \), we have

\[
x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\]

\[
< A(x_n - \xi)^2
\]

\[
< \frac{1}{A} \left[A(x_1 - \xi)\right]^{2^{n+1}},
\]

which proves the statement for all \( n \). Since \( 0 < x_{n+1} - \xi \) for all \( n \), this shows that \( x_n \to \xi \) as \( n \to \infty \). Let \( g(x) = x - f(x)/f'(x) \). Since \( \xi \) is a root of \( f \), \( g(\xi) = \xi \), and \( x_n \to \xi \), the process amounts to finding a fixed point of \( g \). For \( x \) near \( \xi \),

\[
g'(x) = 1 - \frac{f'(x)^2 - f''(x)f(x)}{f'(x)^2}
\]

\[
= \frac{f(x)f''(x)}{f'(x)^2}
\]

\[
\approx 0.
\]

**Theorem 81.** [Exercise 5.26] Suppose that \( f : [a, b] \to \mathbb{R} \) is a differentiable function with \( f(a) = 0 \). Let \( A \) be a real number such that \( |f'(x)| \leq A|f(x)| \) for all \( x \in [a, b] \). Then \( f = 0 \).
Proof. Let \( x_0 \in [a, b], \) \( M_0 = \sup_{a \leq x \leq x_0} |f(x)|, \) and \( M_1 = \sup_{a \leq x \leq x_0} |f'(x)|. \) By the mean value theorem, there exists a point \( c \in (a, x_0) \) such that

\[
\frac{f(x_0)}{x_0 - a} = f'(c)
\]

\[
|f(x_0)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.
\]

Suppose that \( x_0 > a \) and let \( x \in (a, x_0) \). By the mean value theorem, there exists a point \( c \in (a, x) \) such that

\[
\frac{f(x)}{x - a} = f'(c)
\]

\[
|f(x)| \leq M_1(x - a)
\]

\[
\leq M_1(x_0 - a) \leq A(x_0 - a)M_0.
\]

Since \( f(a) = 0 \), we have \( |f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0 \) for all \( x \in [a, b] \). Suppose that \( A(x_0 - a) < 1 \); then \( M_0 = 0 \) for otherwise \( A(x_0 - a)M_0 < M_0 \) is a lower bound of \( |f(x)| \) in \( [a, x_0] \), which contradicts the definition of \( M_0 \). Therefore, if \( x_0 > a \) is small enough, then \( f(x) = 0 \) for all \( x \in [a, x_0] \). Now divide the interval \( [a, b] \) into \( n \) closed intervals \([a, p_1], [p_1, p_2], \ldots, [p_n, b]\) where \( n \) is the smallest integer with \( n(x_0 - a) \geq b - a \), and \( p_k = a + k(x_0 - a) \). We have shown that \( f \) is zero on \([a, x_0] = [a, p_1]\); since \( f(p_1) = 0 \), applying the argument on \([p_1, p_2]\) shows that \( f \) is zero on \([p_1, p_2]\), and so on. \( \square \)

**Theorem 82.** [Exercise 5.27] Let \( R \) be a rectangle in the plane given by \( a \leq x \leq b \) and \( \alpha \leq y \leq \beta \) for \((x, y) \in R \). Let \( \phi : R \to \mathbb{R} \) be a function defined on the rectangle. A **solution** of the initial-value problem

\[
y' = \phi(x, y), \quad y(a) = c \quad \text{where } \alpha \leq c \leq \beta
\]

is by definition a differentiable function \( f : [a, b] \to [\alpha, \beta] \) such that \( f(a) = c \) and \( f'(x) = \phi(x, f(x)) \) for all \( x \in [a, b] \). Suppose that there is a constant \( A \) such that

\[
|\phi(x, y_2) - \phi(x, y_1)| \leq A |y_2 - y_1|
\]

whenever \((x, y_1) \in R \) and \((x, y_2) \in R \). Then the problem has at most one solution.

**Proof.** Let \( f, g \) be two solutions of the initial-value problem, and let \( h : [a, b] \to \mathbb{R} \) be given by \( h(x) = f(x) - g(x) \). Then

\[
|h'(x)| = |f'(x) - g'(x)|
\]

\[
= |\phi(x, f(x)) - \phi(x, g(x))|
\]

\[
\leq A |f(x) - g(x)|
\]

\[
= A |h(x)|
\]

for all \( x \in [a, b] \). Since \( h(a) = 0 \), by Theorem 81 \( h = 0 \) and \( f = g \). \( \square \)
Chapter 6. The Riemann-Stieltjes Integral

Theorem 83. [Exercise 6.1] Suppose \( \alpha : [a,b] \to \mathbb{R} \) is increasing, \( a \leq x_0 \leq b \), \( \alpha \) is continuous at \( x_0 \), \( f(x_0) = 1 \), and \( f(x) = 0 \) if \( x \neq x_0 \). Then \( f \in \mathcal{R}(\alpha) \) and \( \int_a^b f \, d\alpha = 0 \).

Proof. By Theorem 6.10, \( f \in \mathcal{R}(\alpha) \) since \( f \) has only one point of discontinuity. Also, since \( L(P,f,\alpha) = 0 \) for all partitions \( P \), \( \int_a^b f \, d\alpha = 0 \).

Theorem 84. [Exercise 6.2] Suppose \( f : [a,b] \to \mathbb{R} \) is a continuous function, \( f \geq 0 \), and \( \int_a^b f(x) \, dx = 0 \). Then \( f = 0 \).

Proof. Suppose that \( f \neq 0 \); we can choose \( x_0 \in (a,b) \) such that \( f(x_0) > 0 \), for \( f \) cannot be nonzero only at its endpoints due to continuity. Then there exists a \( \delta > 0 \) such that \( |f(x_0) - f(x)| < f(x_0)/2 \) whenever \( |x_0 - x| < \delta \). In particular, \( f(x) > f(x_0)/2 \) for all \( x \in [x_0 - \gamma, x_0 + \gamma] \), where \( \gamma = \min \{ \delta/2, x_0 - a, b - x_0 \} \). By Theorem 6.12,

\[
\int_a^b f(x) \, dx = \int_a^{x_0-\gamma} f(x) \, dx + \int_{x_0-\gamma}^{x_0+\gamma} f(x) \, dx + \int_{x_0+\gamma}^b f(x) \, dx \\
\geq \int_{x_0-\gamma}^{x_0+\gamma} f(x) \, dx \\
\geq \int_{x_0-\gamma}^{x_0+\gamma} f(x_0)/2 \, dx \\
> 0,
\]

which is a contradiction. Therefore \( f = 0 \).

Theorem 85. [Exercise 6.3] Define three functions \( \beta_1, \beta_2, \beta_3 \) as follows: \( \beta_j(x) = 0 \) if \( x < 0 \), \( \beta_j(x) = 1 \) if \( x > 0 \) for \( j = 1, 2, 3 \); and \( \beta_1(0) = 0, \beta_2(0) = 1, \beta_2(0) = \frac{1}{2} \). Let \( f \) be a bounded function on \([-1,1]\).

1. \( f \in \mathcal{R}(\beta_1) \) if and only if \( f(0+) = f(0) \), and then \( \int_{-1}^1 f(x) \, d\beta_1 = f(0) \).
2. \( f \in \mathcal{R}(\beta_2) \) if and only if \( f(0-) = f(0) \), and then \( \int_{-1}^1 f(x) \, d\beta_2 = f(0) \).
3. \( f \in \mathcal{R}(\beta_3) \) if and only if \( f \) is continuous at 0.
4. If \( f \) is continuous at 0 then

\[
\int_{-1}^1 f(x) \, d\beta_1 = \int_{-1}^1 f(x) \, d\beta_2 = \int_{-1}^1 f(x) \, d\beta_3 = f(0).
\]
Proof. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon/2$ whenever $0 < x < \delta$. Let $\gamma = \min(1, \delta)/2$ and let $P = \{-1, 0, \gamma, 1\}$ be a partition of $[-1, 1]$. Then
\[
U(P, f, \beta_1) - L(P, f, \beta_1) = \sup_{x \in [0, \gamma]} f(x) - \inf_{x \in [0, \gamma]} f(x) < \varepsilon,
\]
so $f \in R(\beta_1)$. Furthermore,
\[
U(P, f, \beta_1) = \sup_{x \in [0, \gamma]} f(x) \leq f(0) + \frac{\varepsilon}{2},
\]
which shows that $\int_{-1}^{1} f(x) \, d\beta_1 = f(0)$ since $\varepsilon$ was arbitrary. Conversely, suppose that $f \in R(\beta_1)$. Let $\varepsilon > 0$ be given. There exists a partition $P$ of $[-1, 1]$ such that
\[
U(P, f, \beta_1) - L(P, f, \beta_1) = M_i - m_i < \varepsilon
\]
for some $i$ with $x_{i-1} \leq 0 < x_i$, where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. Then whenever $0 < t < x_i$ we have $0 \leq f(t) - m_i < \varepsilon$ and $-\varepsilon < m_i - f(0) \leq 0$ so that $|f(t) - f(0)| < \varepsilon$. This shows that $f(0^+) = f(0)$. The proof is similar for (2) and (3). \hfill \Box

**Theorem 86.** [Exercise 6.4] If $f(x) = 0$ for all irrational $x$ and $f(x) = 1$ for all rational $x$, then $f \notin R$ on $[a, b]$ for any $a < b$.

**Proof.** Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. For all $x < y$ there exist both rational and irrational numbers in $(x, y)$, so $M_i = 1$ and $m_i = 0$ for every $i$. Therefore
\[
U(P, f) - L(P, f) = \sum_{i=1}^{n} \Delta x_i = b - a,
\]
and $f \notin R$ on $[a, b]$. \hfill \Box

**Remark 87.** [Exercise 6.5] Suppose $f$ is a bounded real function on $[a, b]$, and $f^2 \in R$ on $[a, b]$. Does it follow that $f \in R$? Does the answer change if we assume that $f^3 \in R$?

Assume that $a < b$ and let $f(x) = 1$ if $x \in \mathbb{Q}$, $f(x) = -1$ if $x \notin \mathbb{Q}$. Then $f^2 \in R$ with $\int_{a}^{b} f(x)^2 \, dx = b - a$, but $f \notin R$. This disproves the first part of the statement. However, the second statement is true by Theorem 6.11, since $x \mapsto x^{1/3}$ is continuous on any interval in $\mathbb{R}$. 
Theorem 88. [Exercise 6.7] Let \( f : (0, 1] \to \mathbb{R} \) and suppose that \( f \in R \) on \([c, 1]\) for every \( c > 0 \). Define
\[
\int_0^1 f(x) \, dx = \lim_{c \to 0} \int_c^1 f(x) \, dx
\]
if this limit exists (and is finite).

1. If \( f \in R \) on \([0, 1]\), then this definition of the integral agrees with the old one.
2. There exists a function \( f \) such that the above limit exists, although it fails to exist with \(|f|\) in place of \( f \).

Proof. If \( f \in R \) on \([0, 1]\), then by Theorem 6.20,
\[
F(c) = \int_c^1 f(x) \, dx
\]
is continuous on \([0, 1]\). Therefore
\[
\lim_{c \to 0} F(c) = \int_0^1 f(x) \, dx.
\]
\( \square \)

Theorem 89. [Exercise 6.8] Suppose that \( f \in R \) on \([a, b]\) for every \( b > a \) where \( a \) is fixed. Define
\[
\int_a^b f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx
\]
if this limit exists (and is finite). Assume that \( f(x) \geq 0 \) and that \( f \) decreases monotonically on \([1, \infty)\). Then \( \int_1^\infty f(x) \, dx \) converges if and only if \( \sum_{n=1}^\infty f(n) \) converges.

Proof. Suppose that \( \int_1^\infty f(x) \, dx \) converges to \( L \). For every \( \varepsilon > 0 \), there exists a \( M \geq 1 \) such that
\[
\left| \int_1^b f(x) \, dx - L \right| < \varepsilon/2
\]
whenever \( b \geq M \). Then for all \( n \geq m \geq M + 1 \), we have
\[
\int_{m-1}^n f(x) \, dx \leq \int_1^n f(x) \, dx - L + L - \int_{m-1}^{n-1} f(x) \, dx < \varepsilon.
\]
But since \( f \) decreases monotonically on \([1, \infty)\),
\[
0 \leq \sum_{k=m}^n f(k) \leq \sum_{k=m}^n \int_{k-1}^k f(x) \, dx = \int_{m-1}^n f(x) \, dx < \varepsilon,
\]
which shows that \( \sum_{n=1}^\infty f(n) \) converges. Conversely, suppose that \( \sum_{n=1}^\infty f(n) \) converges; we first show that the sequence \( \{ \int_1^n f(x) \, dx \} \) converges. Let \( \varepsilon > 0 \). There exists a
\[ M \geq 1 \text{ such that for all } n \geq m \geq M \text{ we have } 0 \leq \sum_{k=m}^{n} f(k) < \varepsilon. \] Then for all \( m, n \geq M \), assume \( m \leq n \) so that
\[
0 \leq \int_{1}^{n} f(x) \, dx - \int_{1}^{m} f(x) \, dx = \int_{m}^{n} f(x) \, dx = \sum_{k=m}^{n-1} \int_{k}^{k+1} f(x) \, dx \\
\leq \sum_{k=m}^{n-1} f(k) < \varepsilon.
\]

This shows that \( \int_{1}^{i} f(x) \, dx \to L \) for some \( L \geq 0 \), and furthermore, \( \int_{1}^{i} f(x) \, dx \leq L \) for all \( i \geq 1 \) since the sequence is monotonically increasing. Let \( \varepsilon > 0 \) be given; there exists a \( N \geq 1 \) such that \( 0 \leq L - \int_{1}^{i} f(x) \, dx < \varepsilon \) whenever \( i \geq N \). Now for all real \( b \geq N + 1 \),
\[
\int_{1}^{[b]} f(x) \, dx \leq \int_{1}^{b} f(x) \, dx \\
0 \leq L - \int_{1}^{b} f(x) \, dx \leq L - \int_{1}^{[b]} f(x) \, dx < \varepsilon.
\]

This proves that \( \int_{1}^{\infty} f(x) \, dx \) converges to \( L \). \qed

**Theorem 90.** [Exercise 6.9] Suppose that \( F \) and \( G \) are differentiable on \([a, b]\) for every \( b > a \), \( F' = f \in \mathcal{R} \) and \( G' = g \in \mathcal{R} \). If
\[
\lim_{b \to \infty} F(b)G(b)
\]
exists (with a finite value) and
\[
\int_{a}^{\infty} f(x)G(x) \, dx
\]
converges, then
\[
\int_{a}^{\infty} F(x)g(x) \, dx = \lim_{b \to \infty} F(b)G(b) - F(a)G(a) - \int_{a}^{\infty} f(x)G(x) \, dx.
\]

**Proof.** For every \( b > a \),
\[
\int_{a}^{b} F(x)g(x) \, dx = F(b)G(b) - F(a)G(a) - \int_{a}^{b} f(x)G(x) \, dx.
\]
The result follows from Theorem 4.4. \qed
Theorem 91. [Exercise 6.10] Let $p$ and $q$ be positive real numbers such that
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

(1) If $u \geq 0$ and $v \geq 0$, then
\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q},
\]
with equality if $u^p = v^q$.

(2) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and
\[
\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,
\]
then
\[
\int_a^b fg \, d\alpha \leq 1.
\]

(3) If $f$ and $g$ are complex functions in $\mathcal{R}(\alpha)$, then
\[
\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.
\]

Proof. We have
\[
uv = (u^p)^{1/p}(v^q)^{1/q}
\]
\[
= \exp \left( \frac{1}{p} \log u^p \right) \exp \left( \frac{1}{q} \log v^q \right)
\]
\[
= \exp \left( \frac{1}{p} \log u^p + \frac{1}{q} \log v^q \right)
\]
\[
\leq \frac{1}{p} \exp (\log u^p) + \frac{1}{q} \exp (\log v^q)
\]
\[
= \frac{u^p}{p} + \frac{v^q}{q}
\]
since $1/q = 1 - 1/p$ and $\exp$ is convex. If $u^p = v^q$, then
\[
uv = (u^p)^{1/p}(v^q)^{1/q}
\]
\[
= (u^p)^{1/p + 1/q}
\]
\[
= u^p
\]
\[
= \frac{u^p}{p} + \frac{v^q}{q}.
\]
This proves (1). Let \( f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha) \) with \( f \geq 0, g \geq 0 \) and
\[
\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha.
\]
Then (on \([a, b]\))
\[
fg \leq \frac{f^p + g^q}{p + q},
\]
\[
\int_a^b fg \, d\alpha \leq \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha
\]
\[
= 1,
\]
which proves (2). Now suppose that \( f \) and \( g \) are functions in \( \mathcal{R}(\alpha) \). Let
\[
A = \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p},
\]
\[
B = \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q},
\]
so that
\[
\int_a^b \left( \frac{|f|}{A} \right)^p \, d\alpha = 1 = \int_a^b \left( \frac{|g|}{B} \right)^q \, d\alpha
\]
assuming that \( A, B > 0 \). Applying (2) gives
\[
\int_a^b \frac{|f| |g|}{AB} \, d\alpha \leq 1,
\]
and then
\[
\left| \int_a^b fg \, d\alpha \right| \leq \int_a^b |f| |g| \, d\alpha
\]
\[
\leq AB
\]
\[
= \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q},
\]
which proves (3). \(\square\)

**Theorem 92.** [Exercise 6.11] Let \( \alpha \) be a fixed increasing function on \([a, b]\). For \( u \in \mathcal{R}(\alpha) \), define
\[
\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.
\]
Suppose \( f, g, h \in \mathcal{R}(\alpha) \). Then
\[
\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.
\]
Proof. On \([a, b]\) we have

\[
(f - h)^2 = (f - g + g - h)^2
= (f - g)^2 + 2(f - g)(g - h) + (g - h)^2
\]

\[
\int_a^b |f - h|^2 \, d\alpha = \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b (f - g)(g - h) \, d\alpha + \int_a^b |g - h|^2 \, d\alpha
\]

\[
\leq \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b (f - g)(g - h) \, d\alpha + \int_a^b |g - h|^2 \, d\alpha.
\]

Applying Theorem 91 gives

\[
\|f - h\|^2 \leq \|f - g\|^2 + 2 \left( \int_a^b |f - g|^2 \, d\alpha \right)^{1/2} \left( \int_a^b |g - h|^2 \, d\alpha \right)^{1/2} + \|g - h\|^2
\]

\[
= \|f - g\|^2 + 2 \|f - g\|_2 \|g - h\|_2 + \|g - h\|^2
\]

\[
= (\|f - g\|_2 + \|g - h\|_2)^2,
\]

which completes the proof.

Theorem 93. [Exercise 6.12] Suppose \(f \in \mathcal{R}(\alpha)\) and \(\varepsilon > 0\). Then there exists a continuous function \(g\) on \([a, b]\) such that \(\|f - g\|_2 < \varepsilon\).

Proof. Let \(M = \sup f(x)\) and \(m = \inf f(x)\) over \(x \in [a, b]\), and assume that \(M \neq m\) for otherwise \(f\) is constant and the result follows by setting \(g = f\). Let \(P = \{x_0, \ldots, x_n\}\) be a partition of \([a, b]\) such that \(U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon^2/(M - m)\). Define

\[
g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)
\]

for \(x_{i-1} \leq t \leq x_i\); \(g\) is continuous at each \(x_i\). For each \(i\), let \(M_i = \sup f(x)\) and \(m_i = \inf f(x)\), over \(x \in [x_{i-1}, x_i]\). We can rewrite \(g\) as

\[
g(t) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{\Delta x_i} (t - x_{i-1}),
\]

which shows that

\[
m \leq m_i \leq g(x) \leq M_i \leq M
\]
for all \( x \in [a, b] \). Then

\[
\|f - g\|_2^2 = \int_a^b |f(x) - g(x)|^2 \, d\alpha \\
= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x) - g(x)|^2 \, d\alpha \\
= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |M_i - m_i|^2 \, d\alpha \\
= \sum_{i=1}^n |M_i - m_i|^2 \, \Delta \alpha_i \\
\leq (M - m) \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \\
= (M - m)(U(P, f, \alpha) - L(P, f, \alpha)) \\
< \varepsilon^2,
\]

which completes the proof. \( \square \)

**Theorem 94.** [Exercise 6.13] Define

\[
f(x) = \int_x^{x+1} \sin(t^2) \, dt.
\]

(1) \(|f(x)| < 1/x\) if \( x > 0 \).
(2) \(2xf(x) = \cos(x^2) - \cos((x + 1)^2) + r(x)\) where \(|r(x)| < c/x\) and \( c \) is a constant.
(3) \(\limsup_{x \to \infty} xf(x) = 1\) and \(\liminf_{x \to \infty} xf(x) = -1\).
(4) \(\int_0^\infty \sin(t^2) \, dt\) converges.

**Proof.** Let \( x > 0 \). By Theorem 6.8,

\[
u \mapsto \frac{\sin(u)}{2\sqrt{u}}
\]

is Riemann-integrable on \([x^2, (x + 1)^2]\). Let \( \varphi : [x, x + 1] \to [x^2, (x + 1)^2] \) be given by \( t \mapsto t^2 \). Since \( \varphi \) strictly increasing and onto, applying Theorem 6.19 gives

\[
\int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} \, du = \int_x^{x+1} \sin(t^2) \, dt = f(x).
\]
Let \( F(u) = -\cos u \) and \( G(u) = 1/(2\sqrt{u}) \) so that \( F'(u) = \sin u \) and \( G'(u) = -1/(4u^{3/2}) \). By Theorem 6.22,

\[
\begin{align*}
 f(x) & = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du \\
 & \leq \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} \, du \\
 & = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\
 & = \frac{\cos(x^2) + 1}{2x} - \frac{\cos((x+1)^2) + 1}{2(x+1)} \\
 & \leq \frac{1}{x} - \frac{1}{x+1} \\
 & < \frac{1}{x},
\end{align*}
\]

and similarly replacing \( \cos u \) with 1 gives \(-1/x < f(x)\). This proves (1). For (2),

\[
(*) \quad 2xf(x) = \cos(x^2) - \cos((x+1)^2) + r(x)
\]

where

\[
r(x) = \frac{1}{x+1} \cos((x+1)^2) - x \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} \, du.
\]

Furthermore,

\[
|r(x)| \leq \frac{1}{x+1} + x \int_{x^2}^{(x+1)^2} \frac{1}{2u^{3/2}} \, du \\
= \frac{1}{x+1} + x \left( \frac{1}{x} - \frac{1}{x+1} \right) \\
= \frac{2}{1+x} \\
< \frac{2}{x},
\]

since \( 2x < 2 + 2x \). Then equation (*) shows (3). The integral \( \int_0^\infty \sin(t^2) \, dt \) converges if \( \int_1^\infty \sin(t^2) \, dt \) converges. As in (1) we have for all \( b > 1 \),

\[
\int_1^b \sin(t^2) \, dt = \int_1^{b^2} \frac{\sin(u)}{2\sqrt{u}} \, du \\
\]

and

\[
\int_1^{b^2} \frac{\sin(u)}{2\sqrt{u}} \, du = -\frac{\cos(b^2)}{2b} + \frac{\cos 1}{2} - \int_1^{b^2} \frac{\cos u}{4u^{3/2}} \, du.
\]
Since \( \int_1^\infty 1/(4u^{3/2}) \, du \) converges, applying Theorem 90 shows that \( \int_0^\infty \sin(t^2) \, dt \) converges. \( \square \)

**Theorem 95.** [Exercise 6.15] Suppose that \( f : [a, b] \to \mathbb{R} \) is a continuously differentiable function with \( f(a) = f(b) = 0 \), and

\[
\int_a^b f(x)^2 \, dx = 1.
\]

Then

\[
\int_a^b x f(x) f'(x) \, dx = -\frac{1}{2}
\]

and

\[
\left( \int_a^b f'(x)^2 \, dx \right) \left( \int_a^b x^2 f(x)^2 \, dx \right) > \frac{1}{4}.
\]

**Proof.** Let \( F(x) = f(x) \) and \( G(x) = xf(x) \) so that \( F'(x) = f'(x) \) and \( G'(x) = xf'(x) + f(x) \). By Theorem 6.22,

\[
\int_a^b x f(x) f'(x) \, dx = -\int_a^b f(x)[xf'(x) + f(x)] \, dx
\]

\[
= -\int_a^b f(x)^2 \, dx - \int_a^b xf(x)f'(x) \, dx
\]

\[
= -\frac{1}{2}.
\]

By Theorem 91 we have

\[
\frac{1}{4} = \left| \int_a^b [f'(x)] [xf(x)] \, dx \right|^2 \leq \left( \int_a^b |f'(x)|^2 \, dx \right) \left( \int_a^b |xf(x)|^2 \, dx \right).
\]

\( \square \)

**Theorem 96.** [Exercise 6.16] For \( 1 < s < \infty \), define

\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.
\]

1. \( \zeta(s) = s \int_1^\infty \frac{x}{x^{s+1}} \, dx \).
2. \( \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x-x}{x^{s+1}} \, dx \).
Proof. For every positive integer \( N \),

\[
\begin{align*}
  s \int_1^N \frac{\lfloor x \rfloor}{x^{s+1}} \, dx &= s \sum_{n=1}^{N-1} \int_n^{n+1} \frac{\lfloor x \rfloor}{x^{s+1}} \, dx \\
  &= s \sum_{n=1}^{N-1} n \int_n^{n+1} \frac{1}{x^{s+1}} \, dx \\
  &= \sum_{n=1}^{N-1} n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\
  &= \sum_{n=1}^{N-1} \left( \frac{1}{n^{s-1}} - \frac{n+1}{(n+1)^s} + \frac{1}{(n+1)^s} \right) \\
  &= \sum_{n=1}^{N-1} \left( \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right) + \sum_{n=2}^N \frac{1}{n^s} \\
  &= 1 - \frac{1}{N^{s-1}} + \sum_{n=2}^N \frac{1}{n^s} \\
  &= \sum_{n=1}^N \frac{1}{n^s} - \frac{1}{N^{s-1}}
\end{align*}
\]

so that

\[
  s \int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}} \, dx = \sum_{n=1}^\infty \frac{1}{n^s}
\]

since \( s - 1 > 0 \). For (2), we have

\[
\begin{align*}
  \frac{s}{s-1} - s \int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}} \, dx &= \frac{s}{s-1} - s \int_1^N \frac{1}{x^s} \, dx + s \int_1^N \frac{\lfloor x \rfloor}{x^{s+1}} \, dx \\
  &= \sum_{n=1}^N \frac{1}{n^s} + \left( \frac{s}{s-1} \right) \frac{1}{N^{s-1}} - \frac{1}{N^{s-1}}
\end{align*}
\]

and again,

\[
\begin{align*}
  \frac{s}{s-1} - s \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}} \, dx &= \sum_{n=1}^\infty \frac{1}{n^s}
\end{align*}
\]
if $s > 1$. In fact, the integral in (2) converges for all $s > 0$ since

$$
\int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}} \, dx \leq \int_1^N \frac{1}{x^{s+1}} \, dx = \frac{1}{s} \left( 1 - \frac{1}{N^s} \right).
$$

\[\blacksquare\]

**Lemma 97.** Suppose that $f \in \mathcal{R}$ on $[a, b]$ and let $P$ be a partition of $[a, b]$. Let $c$ be a real number. If $U(P^*, f, \alpha) \geq c$ for every refinement $P^*$ of $P$, then $\int_a^b f \, d\alpha \geq c$. If $L(P^*, f, \alpha) \leq c$ for every refinement $P^*$ of $P$, then $\int_a^b f \, d\alpha \leq c$.

**Proof.** Let $\varepsilon > 0$. There exists a partition $P'$ of $[a, b]$ such that $U(P', f, \alpha) < \int_a^b f \, d\alpha + \varepsilon$. Let $P^* = P \cup P'$; since $P^*$ is a refinement of $P$, we have

$$
2 \leq U(P^*, f, \alpha) \leq U(P', f, \alpha) < \int_a^b f \, d\alpha + \varepsilon,
$$

which completes the proof since $\varepsilon > 0$ was arbitrary. The case for the lower sums is analogous. \[\blacksquare\]

**Theorem 98.** [Exercise 6.17] Suppose $\alpha$ increases monotonically on $[a, b]$, $g$ is continuous, and $g(x) = G'(x)$ for all $x \in [a, b]$. Then

$$
\int_a^b \alpha(x)g(x) \, dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x) \, d\alpha.
$$

**Proof.** Let $\varepsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that $U(P, g) - L(P, g) < \varepsilon$. Applying the mean value theorem gives points $t_i \in (x_{i-1}, x_i)$ such that $g(t_i) \Delta x_i = G(x_i) - G(x_{i-1})$. Then

$$
\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i = \sum_{i=1}^n \alpha(x_i) [G(x_i) - G(x_{i-1})]
$$

$$
= \sum_{i=1}^{n+1} \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1})
$$

$$
(*) \quad = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i
$$

and
\[ \sum_{i=1}^{n} |g(x_i) - g(t_i)| \Delta x_i < \varepsilon \]
by Theorem 6.7 so that
\[
\left| \sum_{i=1}^{n} \alpha(x_i) g(x_i) \Delta x_i - \sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i \right| = \left| \sum_{i=1}^{n} \alpha(x_i) [g(x_i) - g(t_i)] \Delta x_i \right|
\leq \sum_{i=1}^{n} |\alpha(x_i) [g(x_i) - g(t_i)]| \Delta x_i
\leq M \varepsilon
\]
where \( M = \sup \alpha(x) \) over \( x \in [a, b] \). From (8) we have
\[
\sum_{i=1}^{n} \alpha(x_i) g(x_i) \Delta x_i \leq G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i + M \varepsilon
\]
\[
L(P, \alpha g) + L(P, G, \alpha) \leq G(b)\alpha(b) - G(a)\alpha(a) + M \varepsilon
\]
and similarly
\[
G(b)\alpha(b) - G(a)\alpha(a) - M \varepsilon \leq U(P, \alpha g) + U(P, G, \alpha).
\]
But these two inequalities are true for any refinement of \( P \), so by Theorem 97,
\[
S - M \varepsilon \leq \int_{a}^{b} \alpha(x) g(x) \, dx = \int_{a}^{b} \alpha(x) g(x) \, dx \leq S + M \varepsilon
\]
where
\[
S = G(b)\alpha(b) - G(a)\alpha(a) - \int_{a}^{b} G(x) \, d\alpha.
\]
Since \( \varepsilon \) was arbitrary, the result follows. \( \square \)

**Theorem 99.** [Exercise 6.18] Let \( \gamma_1, \gamma_2, \gamma_3 \) be curves in the complex plane, defined on \([0, 2\pi]\) by
\[
\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi t \sin(1/t)}.
\]
(1) \( \gamma_1, \gamma_2 \) are rectifiable. \( \gamma_1 \) has length \( 2\pi \) and \( \gamma_2 \) has length \( 4\pi \).
(2) \( \gamma_3 \) is not rectifiable.

**Proof.** Applying Theorem 6.27 shows that
\[
\Lambda(\gamma_1) = \int_{0}^{2\pi} |ie^{it}| \, dt = 2\pi
\]
and

\[ \Lambda(\gamma_1) = \int_0^{2\pi} |2ie^{2it}| \, dt = 4\pi. \]

Let \( P = \{x_{2n+1}, \ldots, 2/\pi\} \) with

\[ x_k = \frac{2}{(2k + 1)\pi} \]

so that

\[ \Lambda(P, \gamma_3) = \sum_{k=1}^{2n+1} \left| e^{2\pi i x_k \sin(1/x_k)} - e^{2\pi i x_{k-1} \sin(1/x_{k-1})} \right| \]

\[ \geq \sum_{k=1}^{n} \left| e^{4i/(4k+1)} - e^{-4i/(4k-1)} \right| \]

\[ = \sum_{k=1}^{n} \sqrt{2 - 2 \cos \left( \frac{4}{4k+1} + \frac{4}{4k-1} \right)} \]

\[ \to \infty \]

as \( n \to \infty \) since \( \sqrt{2 - 2 \cos x} = x + O(x^3) \) and

\[ \sum_{k=1}^{\infty} \left( \frac{4}{4k+1} + \frac{4}{4k-1} \right) \]

diverges. This shows that \( \Lambda(\gamma_3) = +\infty \) and therefore \( \gamma_3 \) is not rectifiable. \[ \square \]

**Theorem 100.** [Exercise 6.19] Let \( \gamma_1 : [a, b] \to \mathbb{R}^k \) be a curve and let \( \phi : [c, d] \to [a, b] \) be a continuous bijection such that \( \phi(c) = a \). Define \( \gamma_2 = \gamma_1 \circ \phi \). Then:

1. \( \gamma_2 \) is an arc if and only if \( \gamma_1 \) is an arc.
2. \( \gamma_2 \) is a closed curve if and only if \( \gamma_1 \) is a closed curve.
3. \( \gamma_2 \) is rectifiable if and only if \( \gamma_1 \) is rectifiable, and in that case \( \gamma_1, \gamma_2 \) have the same length.

**Proof.** (1) is clear since the composition of injections is also an injection (\( \phi, \phi^{-1} \) are both injective). (2) is clear since \( \phi \) is monotonically increasing and \( \phi(d) = b \). For (3), suppose that \( \gamma_1 \) is rectifiable. Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([c, d]\). Define \( P' = \{\phi(x_0), \ldots, \phi(x_n)\} \); This is a well-defined partition of \([a, b]\), for \( \phi \) must be monotonically
Then for all $x$ increasing. Then
\[
\Lambda(P, \gamma_2) = \sum_{i=1}^{n} |\gamma_1(\phi(x_i)) - \gamma_1(\phi(x_{i-1}))|
\]
\[
= \Lambda(P', \gamma_1)
\leq \Lambda(\gamma_1).
\]
Since this holds for all partitions, we have $\Lambda(\gamma_2) \leq \Lambda(\gamma_1)$ which shows that $\gamma_2$ is rectifiable. Noting that $\gamma_1 = \gamma_2 \circ \phi^{-1}$, the same argument proves that $\Lambda(\gamma_1) \leq \Lambda(\gamma_2)$. □

Chapter 7. Sequences and Series of Functions

**Theorem 101.** [Exercise 7.1] Every uniformly convergent sequence of bounded functions is uniformly bounded.

*Proof.* Let $f_n \to f$ uniformly on $E$, where each $f_n$ is bounded. That is, for each $n$, $M_n = \sup_{x \in E} |f_n(x)|$ is finite. Choose an integer $N$ such that $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in E$. In particular,
\[
|f(x)| \leq |f_N(x) - f(x)| + |f_N(x)|
< M_N + 1
\]
for all $x \in E$, and
\[
|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|
< M_N + 2
\]
for all $n \geq N$. Take $M = \max \{M_1, \ldots, M_{N-1}, M_N + 2\}$ so that $|f_n(x)| \leq M$ for all $n \geq 1$. This completes the proof. □

**Theorem 102.** [Exercise 7.2] If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set $E$, then $\{f_n + g_n\}$ converges uniformly on $E$. Furthermore, if $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then $\{f_n g_n\}$ converges uniformly on $E$.

*Proof.* Let $f_n \to f$ and $g_n \to g$ uniformly on $E$. For any $\varepsilon > 0$, there exist integers $N_1, N_2$ such that for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon/2$ whenever $n \geq N_1$ and $|g_n(x) - g(x)| < \varepsilon/2$ whenever $n \geq N_2$. Then $|f_n(x) - f(x) + g_n(x) - g(x)| < \varepsilon$ whenever $x \in E$ and $n \geq \max(N_1, N_2)$, which shows that $f_n + g_n \to f + g$ uniformly on $E$. Now suppose that $\{f_n\}, \{g_n\}$ are sequences of bounded functions, so that $f, g$ are bounded. Let $\varepsilon > 0$ be given. Choose $N_1, N_2$ such that for all $x \in E$, $|f_n(x) - f(x)| < \sqrt{\varepsilon}$ whenever $n \geq N_1$ and $|g_n(x) - g(x)| < \sqrt{\varepsilon}$ whenever $n \geq N_2$. Then for all $x \in E$ and $n \geq \max(N_1, N_2)$,
\[
|[f_n(x) - f(x)][g_n(x) - g(x)]| < \varepsilon,
\]
which shows that \((f_n - f)(g_n - g) \to 0\) uniformly on \(E\). Since \(f, g\) are bounded, 
\(f(g_n - g) \to 0\) and \(g(f_n - f) \to 0\) uniformly on \(E\), so that 
\[
f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f)
\]
\[
\to 0
\]
uniformly on \(E\). \(\square\)

**Theorem 103.** [Exercise 7.3] Let \(f_n(x) = x\) and \(g_n(x) = 1/n\); \(f_n \to x\) and \(g_n \to 0\) uniformly on \(\mathbb{R}\), but \(\{f_n g_n\}\) does not converge uniformly.

*Proof.* Choose \(\varepsilon = 1\) and let \(N\) be an integer. Then \((f_n g_n)(N) \geq 1 = \varepsilon\) for all \(n \geq N\), which shows that \(\{f_n g_n\}\) does not converge uniformly. \(\square\)

**Example 104.** [Exercise 7.4] Consider 
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}.
\]
For what values of \(x\) does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is \(f\) continuous wherever the series converges? Is \(f\) bounded?

- The series does not converge when \(x = 0\), and is undefined when \(x = -1/n^2\) for any integer \(n \geq 1\). However, it converges absolutely for all other \(x\).
- The series converges uniformly on a set \(E\) if and only if \(0, -1, -1/2^2, -1/3^2, \ldots\) are all interior points of \(E^c\).
- \(f\) is continuous and bounded on any set where it converges uniformly.

**Example 105.** [Exercise 7.5] Let 
\[
f_n(x) = \begin{cases} 
0 & \text{for } x < \frac{1}{n+1}, \\
\sin^2 \frac{\pi}{x} & \text{for } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\
0 & \text{for } \frac{1}{n} < x.
\end{cases}
\]
For any \(x\), there exists a \(N\) such that \(1/n < x\) for all \(n \geq N\); this shows that \(f_n \to 0\). Choose \(\varepsilon = 1\); then for all \(N\) we have 
\[
f_N \left(\frac{2N + 1}{2N(N + 1)}\right) = \sin^2(2N(N + 1)\pi/(2N + 1)) = 1,
\]
which shows that \(\{f_n\}\) does not converge uniformly. Now consider the series \(\sum f_n(x)\). For any \(x\) there are only finitely many non-zero terms, so that the series converges absolutely for all \(x\). Again, the series fails to converge uniformly.
Theorem 106. [Exercise 7.6] The series

\[ (*) \quad \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \left( \frac{x^2}{n^2} + \frac{1}{n} \right) \]

converges uniformly in every bounded interval, but does not converge absolutely for any value of \( x \).

Proof. Let \( I \) be a bounded interval and let \( M = \sup_{x \in I} |x| \). By Theorem 3.43, \( \sum (-1)^n / n \) converges, and \( \sum (-1)^n x^2 / n^2 \) converges (absolutely) for all \( x \). Therefore \( (*) \) converges, and it remains to show that the convergence is uniform. Let \( \varepsilon > 0 \) be given and choose \( N_1 \) such that

\[ \left| \sum_{k=m}^{n} (-1)^k \frac{1}{n} \right| < \varepsilon / 2 \]

whenever \( n \geq m \geq N_1 \). Also choose \( N_2 \) such that

\[ \sum_{k=m}^{n} M^2 / n^2 < \varepsilon / 2 \]

whenever \( n \geq m \geq N_2 \). Then for all \( n \geq m \geq \max(N_1, N_2) \) and all \( x \in I \),

\[ \left| \sum_{k=m}^{n} (-1)^k \left( \frac{x^2}{n^2} + \frac{1}{n} \right) \right| \leq \left| \sum_{k=m}^{n} (-1)^k \frac{1}{n} \right| + \sum_{k=m}^{n} \frac{x^2}{n^2} \]

\[ \leq \left| \sum_{k=m}^{n} (-1)^k \frac{1}{n} \right| + \sum_{k=m}^{n} \frac{M^2}{n^2} \]

\[ < \varepsilon. \]

This shows that \( (*) \) converges absolutely by Theorem 7.8. That the series does not converge absolutely is clear from the fact that \( \sum 1/n \) diverges. \( \Box \)

Theorem 107. [Exercise 7.7] Let \( f_n : \mathbb{R} \to \mathbb{R} \) be defined for all positive integers \( n \) by

\[ f_n(x) = \frac{x}{1 + nx^2}. \]

Then \( \{ f_n \} \) converges uniformly to a function \( f \), and the equation

\[ (*) \quad f'(x) = \lim_{n \to \infty} f_n'(x) \]

is correct if \( x \neq 0 \) but false if \( x = 0 \).
Proof. Let \( \varepsilon > 0 \) be given and choose an integer \( N \) such that \( N > \frac{1}{\varepsilon^2} \). Let \( n \geq N \) and \( x \in \mathbb{R} \). If \( |x| < \varepsilon \) then
\[
\left| \frac{x}{1 + nx^2} \right| \leq |x| < \varepsilon.
\]
Otherwise,
\[
\left| \frac{x}{1 + nx^2} \right| \leq \frac{1}{nx} < \varepsilon.
\]
This shows that \( f_n \to 0 \) uniformly on \( \mathbb{R} \). For each \( n \) we have
\[
f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.
\]
If \( x \neq 0 \) then \( f'_n(x) \to 0 \) as \( n \to \infty \) so that \( (*) \) is true, but \( f'_n(0) = 1 \) while \( f'(0) = 0 \), which contradicts \( (*) \). \( \Box \)

Theorem 108. [Exercise 7.8] If
\[
I(x) = \begin{cases} 
0 & \text{for } x \leq 0, \\
1 & \text{otherwise},
\end{cases}
\]
if \( \{x_n\} \) is a sequence of distinct points of \( (a, b) \), and if \( \sum |c_n| \) converges, then the series
\[
f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)
\]
converges uniformly on \( [a, b] \). Additionally, \( f \) is continuous for every \( x \neq x_n \).

Proof. Applying Theorem 7.10 shows that the series converges uniformly on \( [a, b] \) since
\[
|c_n I(x - x_n)| \leq |c_n|
\]
for each \( n \) and \( \sum |c_n| \) converges. If \( x \neq x_n \), then there exists a neighborhood \( N \) of \( x \) such that \( N \cap \{x_n\} \) is empty. It is clear from the definition that \( f \) is constant on \( N \), that is, \( f(t) = f(u) \) for all \( t, u \in N \). This shows that \( f \) is continuous at \( x \). \( \Box \)

Theorem 109. [Exercise 7.9] Let \( \{f_n\} \) be a sequence of continuous functions which converges uniformly to a function \( f \) on a set \( E \). Then
\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]
for every sequence of points \( x_n \in E \) such that \( x_n \to x \) and \( x \in E \).
Proof. Let $\varepsilon > 0$ be given. Choose an integer $N_1$ such that $|f_n(t) - f(t)| < \varepsilon/2$ whenever $t \in E$ and $n \geq N_1$. By Theorem 7.12, $f$ is continuous on $E$, so that we may choose a $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon/2$ whenever $|t - x| < \delta$, and choose an integer $N_2$ such that $|x_n - x| < \delta$ whenever $n \geq N_2$. Then for all $n \geq \max(N_1, N_2),

|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.

□

Theorem 110. [Exercise 7.11] Let $\{f_n\}, \{g_n\}$ be sequences in a set $E$. If

1. $\sum f_n$ has uniformly bounded partial sums,
2. $g_n \to 0$ uniformly on $E$, and
3. $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$,

then $\sum f_n g_n$ converges uniformly on $E$.

Proof. Note that $g_k(x) \geq 0$ for all $x \in E$ and $k \geq 1$, since each $\{g_n(x)\}$ is monotonic.

Let $\varepsilon > 0$ be given. Since $\sum f_n$ has uniformly bounded partial sums, we can let $M = \sup_{x \in E} |A_n(x)|$ where $A_n(x)$ denotes the partial sums of $\sum f_n(x)$. Choose an integer $N$ such that $g_N < \varepsilon/(2M)$. Then for all $n \geq m \geq N$ and $x \in E$,

$$\left| \sum_{k=m}^{n} f_k(x) g_k(x) \right| = \left| \sum_{k=m}^{n-1} A_k(x)[g_k(x) - g_{k+1}(x)] + A_n(x)g_n(x) - A_{m-1}(x)g_m(x) \right|
\leq \sum_{k=m}^{n-1} |A_k(x)| |g_k(x) - g_{k+1}(x)| + |A_n(x)g_n(x)| + |A_{m-1}(x)g_m(x)|
\leq M \left( \sum_{k=m}^{n-1} |g_k(x) - g_{k+1}(x)| + g_n(x) + g_m(x) \right)
= 2Mg_m(x)
< \varepsilon.

□

Theorem 111. Let $\{f_n\}$ be a sequence of functions that converge uniformly to $f$ on $[a, \infty)$, where $\lim_{x \to \infty} f_n(x)$ exists for each $n$. Let

$$A_n = \lim_{x \to \infty} f_n(x),$$

then $\{A_n\}$ converges, and

$$\lim_{x \to \infty} f(x) = \lim_{n \to \infty} A_n.$$
Proof. Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to $f$, there exists an integer $N$ such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever every $x \geq a$ and $m, n \geq N$. By Corollary 30, $|A_n - A_m| < \varepsilon$ for all $m, n \geq N$. This shows that $\{A_n\}$ converges to some $A$. Choose an integer $N$ such that $|f(x) - f_N(x)| < \varepsilon/3$ for all $x \geq a$ and $|A_N - A| < \varepsilon/3$. Then choose a $M$ such that $|f_N(x) - A_N| < \varepsilon/3$ for all $x \geq M$, so that
\[
|f(x) - A| \leq |f(x) - f_N(x)| + |f_N(x) - A_N| + |A_N - A|
\]
whenever $x \geq \max(a, M)$. This completes the proof. \[\]

**Theorem 112.** [Exercise 7.12] Let $g, f_n : (0, \infty) \to \mathbb{R}$ be functions Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$. If $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $[0, \infty)$, and
\[
\int_0^\infty g(x) \, dx < \infty,
\]
then
\[
\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty f(x) \, dx,
\]
provided that all improper integrals exist.

**Proof.** Define $F_n : [0, \infty) \to \mathbb{R}$ for each $n$ and $F : [0, \infty) \to \mathbb{R}$ by
\[
F_n(b) = \int_0^b f_n(x) \, dx,
\]
\[
F(b) = \int_0^b f(x) \, dx,
\]
and let $L = \int_0^\infty g(x) \, dx$ for convenience. For every $b$,
\[
\lim_{n \to \infty} F_n(b) = F(b)
\]
by Theorem 7.16, so that $F_n \to F$ pointwise on $[0, \infty)$. We also want to show that convergence is uniform. Let $\varepsilon > 0$ be given. Choose a $M \geq 0$ such that
\[
\int_M^\infty g(x) \, dx = L - \int_0^M g(x) \, dx < \frac{\varepsilon}{4},
\]
and choose an integer $N$ such that
\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2M}
\]
whenever \( n \geq N \) and \( 0 \leq x \leq M \). Then for all \( b \geq M \) and \( n \geq N \),
\[
|F_n(b) - F(b)| = \left| \int_0^b [f_n(x) - f(x)] \, dx \right|
\leq \int_0^b |f_n(x) - f(x)| \, dx
\leq \int_0^M |f_n(x) - f(x)| \, dx + 2 \int_M^\infty g(x) \, dx
< \varepsilon,
\]
while \( |F_n(b) - F(b)| < \varepsilon/4 < \varepsilon \) trivially when \( b < M \). The result then follows from applying Theorem 113 on \( \{F_n\} \).

\[ \Box \]

**Theorem 113.** [Exercise 7.13] Let \( \{f_n\} \) be a sequence of monotonically increasing functions on \( \mathbb{R} \) with \( 0 \leq f_n(x) \leq 1 \) for all \( x \) and all \( n \).

1. There is a function \( f \) and a sequence \( \{n_k\} \) such that
   \[ f(x) = \lim_{k \to \infty} f_{n_k}(x) \]
   for every \( x \in \mathbb{R} \).
2. If \( f \) is continuous, then \( f_{n_k} \to f \) uniformly on compact sets.

**Proof.** By Theorem 7.23, there exists a subsequence of functions \( \{f_{n_k}\} \) such that \( \{f_{n_k}(r)\} \) converges to some \( f(r) \) for all \( r \in \mathbb{Q} \). For all \( x \in \mathbb{R} \), define
   \[ f(x) = \sup_{r \leq x, r \in \mathbb{Q}} f(r). \]
Let \( x \in \mathbb{R} \setminus \mathbb{Q} \) and suppose that \( f \) is continuous at \( x \). Let \( L = \lim_{k \to \infty} f_{n_k}(x) \); we want to show that \( f(x) = L \). For every rational \( r \leq x \) we have \( f_{n_k}(r) \leq f_{n_k}(x) \) and therefore \( f(r) \leq L \) by taking \( k \to \infty \). This shows that \( f(x) \leq L \). Suppose that \( f(x) < L \), and choose a \( \varepsilon > 0 \) with \( f(x) < f(x) + \varepsilon < L \). Choose a \( \delta > 0 \) such that \( |f(x) - f(t)| < \varepsilon \) whenever \( |x - t| < \delta \). If \( r \in \mathbb{Q} \) with \( x < r < x + \delta \), then \( f(r) < L \). But
   \[ L = \lim_{k \to \infty} f_{n_k}(x) \leq \lim_{k \to \infty} f_{n_k}(r) = f(r) < L, \]
which is a contradiction. Therefore \( f(x) = L \). If \( x < y \) then \( f(x) = \lim_{k \to \infty} f_{n_k}(x) \leq \lim_{k \to \infty} f_{n_k}(y) = f(y) \); by Theorem 4.30, \( f \) has at most a countable number of discontinuities \( \{t_i\} \). Applying Theorem 7.23 again to \( \{t_i\} \) produces a subsequence \( \{f_{n_j}\} \) of \( \{f_{n_k}\} \) such that \( f_{n_j}(t_i) \) converges to some \( u_i \) for every \( i \). Redefining \( f(x) \) using the new subsequence \( \{f_{n_j}\} \) proves (1).

For (2), let \( f \) be a continuous function and let \( \{n_k\} \) be a sequence such that \( f(x) = \lim_{k \to \infty} f_{n_k}(x) \) for every \( x \in \mathbb{R} \). Let \( E \subseteq \mathbb{R} \) be a compact set and let \( \varepsilon > 0 \) be given. By Theorem 4.19, \( f \) is uniformly continuous on \( E \), so there exists a \( \delta > 0 \) such that
\[ |f(x) - f(y)| < \varepsilon/3 \text{ whenever } |x - y| < \delta. \text{ Let } A = \inf E \text{ and } B = \sup E; \text{ construct a set of points } \{x_1, \ldots, x_n\} \text{ where } A = x_1 \leq \cdots \leq x_n = B \text{ and } x_{i+1} - x_i < \delta/2 \text{ for all } 1 \leq i \leq n - 1. \text{ Then for each } 1 \leq i \leq n - 1 \text{ we have}

\[ |f(x_{i+1}) - f(x_i)| = \lim_{k \to \infty} |f_{n_k}(x_{i+1}) - f_{n_k}(x_i)| < \varepsilon/3 \]

and we may choose an integer \( N_i \) such that both \( |f_{n_k}(x_{i+1}) - f_{n_k}(x_i)| < \varepsilon/3 \) and \( |f_{n_k}(x_i) - f(x_i)| < \varepsilon/3 \) whenever \( k \geq N_i \); let \( N = \max \{N_i\} \). Let \( x \in E \) and choose a \( j \) such that \( x \in [x_j, x_{j+1}] \). Then for all \( k \geq N \), since each \( f_n \) is monotonically increasing we have

\[ 0 \leq f_{n_k}(x) - f_{n_k}(x_j) \leq f_{n_k}(x_{j+1}) - f_{n_k}(x) \]

\[ < \varepsilon/3 \]

so that

\[ |f_{n_k}(x) - f(x)| \leq |f_{n_k}(x) - f_{n_k}(x_j)| + |f_{n_k}(x_j) - f(x)| + |f(x) - f(x)| \]

\[ < \varepsilon. \]

This completes the proof. \( \square \)

**Theorem 114. [Exercise 7.15]** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function and let \( f_n(t) = f(nt) \) for \( n = 1, 2, 3, \ldots \). If \( \{f_n\} \) is equicontinuous on \([0, 1]\), then \( f \) is constant on \([0, \infty)\).

**Proof.** Suppose that \( f \) is not constant and without loss of generality, let \( 0 \leq x_1 < x_2 \) with \( f(x_1) < f(x_2) \). Since \( \{f_n\} \) is equicontinuous, there exists a \( \delta > 0 \) such that \( |f(nt) - f(nu)| < [f(x_2) - f(x_1)]/2 \) whenever \( n \geq 1, 0 \leq t, u \leq 1, \) and \( |t - u| < \delta \). Let \( n \) be an integer with

\[ n > \max \left\{ \frac{x_2 - x_1}{\delta}, x_1, x_2 \right\}. \]

Set \( t = x_2/n \) and \( u = x_1/n \); then \( 0 \leq t, u < 1 \) and \( |t - u| = (x_2 - x_1)/n < \delta \) so that

\[ |f(nt) - f(nu)| = f(x_2) - f(x_1) \]

\[ < [f(x_2) - f(x_1)]/2, \]

which is a contradiction. \( \square \)

**Theorem 115. [Exercise 7.16]** Let \( \{f_n\} \) be an equicontinuous sequence of functions on a compact set \( K \). If \( \{f_n\} \) converges pointwise on \( K \), then \( \{f_n\} \) converges uniformly on \( K \).

**Proof.** Let \( \varepsilon > 0 \) be given. There exists a \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \varepsilon \) whenever \( n \geq 1, x, y \in K, \) and \( |x - y| < \delta \). The proof is now almost identical to part (2) of Theorem 113. \( \square \)
Theorem 116. [Exercise 7.18] Let \( \{f_n\} \) be a uniformly bounded sequence of functions which are Riemann-integrable on \([a, b]\), and let

\[
F_n(x) = \int_a^x f_n(t) \, dt
\]

for \( a \leq x \leq b \). Then there exists a subsequence \( \{F_{n_k}\} \) which converges uniformly on \([a, b]\).

Proof. Since \( \{f_n\} \) is uniformly bounded, there exists a \( M > 0 \) such that \( |f_n(t)| < M \) for all \( n \) and \( t \). Let \( \varepsilon > 0 \) be given. Then for all \( |x - y| < \varepsilon/M \) and all \( n \) we have

\[
|F_n(x) - F_n(y)| = \left| \int_y^x f_n(t) \, dt \right|
\leq \int_y^x |f_n(t)| \, dt
\leq M |x - y| < \varepsilon,
\]

which shows that \( \{F_n\} \) is equicontinuous. Clearly, \( \{F_n\} \) is also uniformly bounded. The result follows from Theorem 7.25.

Theorem 117. [Exercise 7.20] If \( f \) is continuous on \([0, 1]\) and if

\[
\int_0^1 f(x)x^n \, dx = 0
\]

for all \( n = 0, 1, 2, \ldots \), then \( f(x) = 0 \) on \([0, 1]\).

Proof. By Theorem 7.26, there exists a sequence of polynomials \( P_n \) such that \( P_n \to f \) uniformly on \([0, 1]\). For each \( n \), write \( P_n(x) = \sum_k a_k x^k \) so that

\[
\int_0^1 f(x)P_n(x) \, dx = \int_0^1 f(x) \sum_k a_k x^k \, dx
= \sum_k a_k \int_0^1 f(x) x^k \, dx
= 0.
\]
Since $f$ is bounded on $[0, 1]$, $fP_n \to f^2$ uniformly on $[0, 1]$ by Theorem 102 and

$$f(x) = \lim_{n \to \infty} P_n(x)$$

$$\int_0^1 f(x)^2 \, dx = \int_0^1 \lim_{n \to \infty} f(x)P_n(x) \, dx$$

$$= \lim_{n \to \infty} \int_0^1 f(x)P_n(x) \, dx$$

$$= 0.$$  

Therefore $f(x)^2 = 0$ on $[0, 1]$.  

**Theorem 118.** [Exercise 7.23] Let $P_0 = 0$, and define, for $n = 0, 1, 2, \ldots$, 

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$ 

Then 

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-1, 1]$.

**Proof.** We have the identity

$$P_{n+1}(x) = P_n(x) + \frac{[|x| + P_n(x)] \, [|x| - P_n(x)]}{2}$$

$$= |x| - P_{n+1}(x) = |x| - P_n(x) - \frac{[|x| + P_n(x)] \, [|x| - P_n(x)]}{2}$$

$$= [|x| - P_n(x)] \left[ 1 - \frac{|x| + P_n(x)}{2} \right].$$

By induction on $n$ we have $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ for all $n$ whenever $|x| \leq 1$. By iteration,

$$|x| - P_n(x) = |x| \prod_{k=0}^{n-1} \left( 1 - \frac{|x| + P_k(x)}{2} \right)$$

$$\leq |x| \prod_{k=0}^{n-1} \left( 1 - \frac{|x|}{2} \right)$$

$$= |x| \left( 1 - \frac{|x|}{2} \right)^n.$$
For \( n \geq 1 \), function \( f(x) = x(1 - x/2)^n \) has derivative
\[
 f'(x) = \left(1 - \frac{x}{2}\right)^n - \frac{nx}{2} \left(1 - \frac{x}{2}\right)^{n-1}
 = \left(1 - \frac{x}{2}\right)^{n-1} \left[1 - \left(\frac{n+1}{2}\right)x\right]
\]
which vanishes at \( x_0 = 2/(n+1) \). This value satisfies \( f(x_0) \leq x_0 \). Since \( f'(x) > 0 \) when \( 0 \leq x < x_0 \) and \( f'(x) < 0 \) when \( x_0 < x \leq 1 \),
\[
 |x| - P_n(x) \leq \frac{2}{n+1}
\]
for all \( |x| \leq 1 \). The result follows taking \( n \) large enough.