Chapter 2. Basic Topology

Theorem 1. *[Exercise 2.9(d)] For any set E, $(E^\circ)^c = \overline{E^c}$.*

Proof. Suppose $x \notin \overline{E^c} = E^c \cup (E^c)'$, i.e. $x \in E$ and $x \notin (E^c)'$. Since x is not a limit point of E^c and $x \notin E^c$, there exists a neighborhood N of x such that $N \cap E^c$ is empty, i.e. $N \subseteq E$. This means $x \in E^\circ$. Then $x \in (E^\circ)^c \Rightarrow x \in \overline{E^c}$, which shows that $(E^\circ)^c \subseteq \overline{E^c}$.

Suppose that $x \in \overline{E^c} = E^c \cup (E^c)'$, i.e. $x \notin E$ or x is a limit point of E^c. If $x \notin E$ then $x \notin E^\circ$, which means $x \in (E^\circ)^c$. If x is a limit point of E^c then for any neighborhood N of x there exists a $y \neq x$ in N such that $y \in E^c \Rightarrow y \notin E$. This shows that x cannot be an interior point of E, so $x \in (E^\circ)^c$. Thus $\overline{E^c} = (E^\circ)^c$. □

Theorem 2. *[Exercise 2.19(b)] If A and B are disjoint open sets, then they are separated.*

Proof. We have $A \cap \overline{B} = A \cap (B \cup B') = A \cap B'$ since $A \cap B$ is empty. Suppose that there exists a $x \in A$ that is a limit point of B. Since A is open, there exists a neighborhood N of x such that $N \subseteq A$. Since x is a limit point of B, there exists a $y \in N$ such that $y \in B$. But then $y \in A$; this is a contradiction for A and B are disjoint. Therefore $A \cap B'$ is empty, and $A \cap \overline{B} = \emptyset$. Similarly, $B \cap \overline{A}$ is empty. This shows that A and B are separated. □

Theorem 3. *[Exercise 2.21] Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $a \in A$, $b \in B$, and define*

$$\mathbf{p}(t) = (1-t)\,\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. Then:

1. *A_0 and B_0 are separated subsets of $\mathbb{R}^1$.*
2. *There exists a $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.*
3. *Every convex subset of $\mathbb{R}^k$ is connected.*

Proof. Let $x \in A_0$ so that $\mathbf{p}(x) \in A$. Since A and B are separated, $\mathbf{p}(x)$ is not a limit point of B and $\mathbf{p}(x) \notin B$. So there exists a neighborhood N of $\mathbf{p}(x)$ such that $N \cap B$ is empty. Consider $N_0 = \mathbf{p}^{-1}(N)$, which is a neighborhood of x. For every $y \in N_0$ we have $\mathbf{p}(y) \in N$ which means $\mathbf{p}(y) \notin B$. But then $y \notin B_0$, so x cannot be a limit point of B_0. This shows that $A_0 \cap \overline{B_0}$ is empty. Similarly, $B_0 \cap \overline{A_0}$ is empty. Hence A_0 and B_0 are separated.

We know that $A_0 \cup B_0 \subseteq (0,1)$. Suppose that $A_0 \cup B_0 = (0,1)$. Then $(0,1)$ is the union of two separated sets by part (1), implying that it is disconnected. This is a contradiction, so $A_0 \cup B_0$ is a proper subset of $(0,1)$ and there exists a $t_0 \in (0,1)$ such that $t_0 \notin A_0$ and $t_0 \notin B_0$, i.e. $\mathbf{p}(t_0) \notin A \cup B$.

Let C be a convex subset of $\mathbb{R}^k$ and suppose that $C = A \cup B$ where A and B are separated. Choose some $\mathbf{a} \in A$ and $\mathbf{b} \in B$. Then there exists a $t_0 \in (0,1)$ such that $(1-t_0)\,\mathbf{a} + t_0\mathbf{b} \notin C$ by statement (2). This contradicts the fact that C is a convex set. Hence C must be connected. □

Theorem 4. *[Exercise 2.23] Every separable metric space has a countable base.*

Proof. Let X be a separable metric space and let Y be a countable dense subset of X. Let $B = \{V_{\alpha,r}\}$ be the collection of all neighborhoods $N_r(\alpha)$ where $\alpha \in Y$ and $r \in \mathbb{Q}$. B is countable since $Y \times \mathbb{Q}$ is countable; we want to show that B is a base for X. Let E be an open set in X. For every $x \in E$, there exists a neighborhood N of x with radius r such that $N \subseteq E$. Let r_1 be some positive rational number less than $r/2$ and let $N_1 = N_{r_1}(x)$. Since x is a limit point of Y, there exists a $y \in N_1$ such that $y \in Y$. Now let $V = N_{r_1}(y)$; since $d(x,y) < r_1$, $x \in V$. Also $V \subseteq N \subseteq E$, since for every $v \in V$, $d(v,x) \leq d(v,y) + d(y,x) < 2r_1 < r$. Since $y \in Y$ and $r_1 \in \mathbb{Q}$, $V \in B$. This shows that B is a countable base for X. □

Theorem 5. *[Exercise 2.24] If X is a metric space in which every infinite subset has a limit point, then X is separable.*

Proof. Fix $\delta > 0$ and choose $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Suppose that this process does not terminate after a finite number of steps. Then we have an infinite set $S = \{x_1, x_2, \ldots\}$ in which $d(x_i, x_j) \geq \delta$ for every $j \neq i$. Suppose that x_0 is a limit point of S. Then there are an infinite number elements $x_i \in S$ such that $d(x_0, x_i) < \delta/2$. But if x_i, x_j are two such elements, $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \delta$, which is a contradiction. Hence S cannot have any limit points. This contradicts the assumption that every infinite subset has a limit point, so the process must terminate after a finite number of steps. Let $S_\delta = \{x_1, x_2, \ldots\}$ be the set of points found by this process for some δ.

The union $C = N_\delta(x_1) \cup N_\delta(x_2) \cup \cdots$ covers X for if $x \in X \setminus C$, then x would have been added to S_δ. Let $D = \bigcup_{n=1}^{\infty} S_{1/n}$; we want to show that D is a countable dense subset of X. That D is countable is clear since each $S_{1/n}$ is finite. Let $x \in X$ and let N be a neighborhood of x with radius r. Let n be a positive integer such that $n > 1/r$. There exists some $S_{1/n} \subseteq D$ and some $s \in S_{1/n}$ such that $N_{1/n}(s)$ contains x, since $\bigcup_{s \in S_{1/n}} N_{1/n}(s)$ covers X. Now $d(s,x) < 1/n < r$, so $s \in N$. Therefore x is a limit point of D. This proves that X is separable. □

Lemma 6. *Let X be a metric space with a countable base. Then X is separable.*

Proof. Let $V = \{V_1, V_2, \dots\}$ be a countable base for X. For every i choose an element $x_i \in V_i$, and let $D = \{x_1, x_2, \dots\}$; D is countable since V is countable. Let $x \in X$ and let N be a neighborhood of x. Then N is the union of a subcollection of V and therefore contains some element from D. This shows that x is a limit point of D, and that D is dense in X. □

Theorem 7. *[Exercise 2.25] Every compact metric space K has a countable base, and K is therefore separable.*

Proof. Let B_n be the collection of all neighborhoods $N_r(\alpha)$ with $r = 1/n$ and $\alpha \in K$. Since B_n is an open cover of K and K is compact, there exists a finite subcover $C_n = \{V_1, V_2, \dots, V_k\} \subset B_n$ that covers K. Let $C = C_1 \cup C_2 \cup \cdots$; C is countable since each C_i is countable. Let E be an open set in K. For every $x \in E$, there exists a neighborhood N of x with radius r such that $N \subseteq E$. Let n be a positive integer such that $n > 2/r$. There exists some neighborhood $N_1 \in C_n$ centered at α such that $x \in N_1$, since C_n covers K. Also, $N_1 \subseteq N \subseteq E$ since for every $y \in N_1$, $d(x, y) \le d(x, \alpha) + d(\alpha, y) < 1/n + 1/n < r$. This shows that C is a countable base for K. Lemma 6 shows that K is separable. □

Theorem 8. *[Exercise 2.26] If X is a metric space in which every infinite subset has a limit point, then X is compact.*

Proof. By Theorem 5, X is separable, and by Theorem 4, X has a countable base $V = \{V_1, V_2, \dots\}$. Let $\{G_\alpha\}$ be an open cover of X. For every $x \in X$, there is some open set G_α such that $x \in G_\alpha$. Since V is a base for X, there exists a $V_i \in V$ with $x \in V_i \subseteq G_\alpha$. This means that there is a countable subcover $\{G_i\}$ of X since each G_α was associated with an element of V. Suppose that no finite subcollection of $\{G_i\}$ covers X. For every positive integer n, let $F_n = (G_1 \cup \cdots \cup G_n)^c$. Since $\{G_1, \dots, G_n\}$ is a finite subcollection, each F_n is nonempty while $\bigcap_{n=1}^{\infty} F_n = \left(\bigcup_{i=1}^{\infty} G_i\right)^c$ is empty since $\{G_i\}$ covers X.

Let $E = \{f_1, f_2, \dots\}$ be a set where each f_i is chosen from F_i. Since E is an infinite subset of X, E has a limit point x. Suppose that $x \notin F_i$ for some i. Since F_i^c is open, there exists a neighborhood N of x with radius r such that $N \cap F_i = \emptyset$. In fact, $N \cap F_j = \emptyset$ for every $j \ge i$ since $F_1 \supseteq F_2 \supseteq \cdots$, and therefore $N \cap E$ is finite. But x is a limit point of E, so $N \cap E$ must be infinite. This is a contradiction, and therefore $x \in F_i$ for all i. Then $x \in \bigcap_{n=1}^{\infty} F_n$ but this is a contradiction for $\bigcap_{n=1}^{\infty} F_n$ is empty. Thus there is a finite subcollection of $\{G_i\}$ that covers X, and X must be compact. □

Chapter 3. Numerical Sequences and Series

Theorem 9. *A sequence $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.*

Proof. Suppose that $\{p_n\}$ converges to p and let $\{p_{n_i}\}$ be a subsequence of $\{p_n\}$. Let $\varepsilon > 0$ be given. Then there exists an integer N such that for every $n \geq N$, $d(p_n, p) < \varepsilon$. Let N' be the smallest i such that $n_i \geq N$. Then for every $i \geq N'$, $d(p_{n_i}, p) < \varepsilon$. Therefore $\{p_{n_i}\}$ converges to p. Conversely, suppose that every subsequence of $\{p_n\}$ converges to p. $\{p_n\}$ is a subsequence of itself, so it converges to p. □

Theorem 10. *Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$. If $s_n \leq t_n$ for $n \geq N$ where N is some constant, if $s_n \to s$, and if $t_n \to t$, then $s \leq t$.*

Proof. Assume $s \neq t$ so that $|t - s| > 0$, for otherwise we are done. Since $s_n \to s$ and $t_n \to t$, $t_n - s_n \to t - s$. There exists a M such that for every $m \geq M$, $|t_m - s_m - (t - s)| < |t - s|$. Whenever $k \geq \max(M, N)$, both $t_k - s_k \geq 0$ and $t_k - s_k - (t - s) < |t - s|$ hold. We know $t - s > 0$ for if $t - s < 0$, then $t_k - s_k < 0$ which is a contradiction. □

Theorem 11. *Let $\{x_n\}$ and $\{s_n\}$ be sequences in $\mathbb{R}$. If $0 \leq x_n \leq s_n$ for $n \geq N$ where N is some constant, and if $s_n \to 0$, then $x_n \to 0$.*

Proof. Let $\varepsilon > 0$ be given. Since $s_n \to 0$, there exists a M such that for every $n \geq M$, $|s_n| < \varepsilon$. Let $N' = \max(M, N)$; then for every $n \geq N'$, $|x_n| \leq s_n \leq \varepsilon$. Therefore $x_n \to 0$. □

Corollary 12. *Let $\{x_n\}, \{s_n\}, \{s'_n\}$ be sequences in $\mathbb{R}$. If $s_n \leq x_n \leq s'_n$ for $n \geq N$ where N is some constant, if $s_n \to s$, and if $s'_n \to s$, then $x_n \to s$.*

Theorem 13. *Let $\{s_n\}, \{t_n\}$ be sequences in a metric space. If $s_n \to s$ and $d(s_n, t_n) \to 0$, then $t_n \to s$.*

Proof. Let $\varepsilon > 0$ be given. There exists a M such that $d(s_n, t_n) < \varepsilon/2$ whenever $n \geq M$, and there exists a N such that $d(s, s_n) < \varepsilon/2$ whenever $n \geq N$. Then for all $n \geq \max(M, N)$ we have

$$\begin{aligned} d(s, t_n) &\leq d(s, s_n) + d(s_n, t_n) \\ &< \varepsilon. \end{aligned}$$

□

Theorem 14. *[Theorem 3.19] If $s_n \leq t_n$ for $n \geq N$ where N is fixed, then*

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n \quad \text{and} \quad \liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n.$$

Proof. Let E_1 be the set of subsequential limits of $\{s_n\}$ and let E_2 be the set of subsequential limits of $\{t_n\}$. Let $L_1 = \limsup_{n\to\infty} s_n$ and $L_2 = \limsup_{n\to\infty} t_n$. If $L_1 = -\infty$ or $L_2 = +\infty$, then there is nothing to prove. Otherwise, $L_1 \in E_1$ and there exists a subsequence $\{s_{n_i}\}$ that converges to L_1. Similarly, some $\{t_{n'_i}\}$ converges to L_2. Let m_1

be the minimum i such that $n_i \geq N$ and let m_2 be the minimum i such that $n'_i \geq N$. Let $M = \max(m_1, m_2)$; then $s_{n_i} \leq t_{n'_i}$ for all $i \geq M$ since $s_n \leq t_n$ whenever $n \geq N$. Theorem 10 proves the required result. The case for $\liminf$ is similar. □

Lemma 15. *Let $S = \{s_n\}$ be a sequence in $\mathbb{R}$ and let E be the set of subsequential limits of $\{s_n\}$. Then $\sup E \in (-\infty, +\infty)$ if and only if S is bounded.*

Proof. Suppose that S is not bounded above, i.e. for every $x \in \mathbb{R}$ there exists a $s_i \in S$ such that $s_i > x$. Let $n_1 = 1$ and suppose that $n_1, \ldots, n_k$ have been chosen. Choose n_{k+1} to be the smallest i such that $i > n_k$ and $s_i > s_{n_k}$. Then the subsequence $\{s_{n_k}\}$ approaches $+\infty$ and hence $\sup E = +\infty$. Similarly, if S is not bounded below then $\sup E = -\infty$. Conversely, if $\sup E = +\infty$ then there exists a subsequence $\{s_{n_k}\}$ such that for every M, $s_{n_k} \geq M + 1 > M$ for some n_k. The case for $\sup E = -\infty$ is similar. Hence S is unbounded. □

Theorem 16. *[Equivalence of* $\limsup$ *definitions.] Let $S = \{s_n\}$ be a sequence in $\mathbb{R}$, let $S_n = \{s_n, s_{n+1}, \ldots\}$ and let E be the set of subsequential limits of $\{s_n\}$. Let $L \in [-\infty, \infty]$. Then the following are equivalent:*

(1) $L = \sup E$.
(2) $L \in E$ *and for every* $x > L$ *there is an integer* N *such that* $n \geq N$ *implies* $s_n < x$.
(3) $L = \lim_{n\to\infty} \sup S_n$.

Furthermore, any L with these properties is unique.

Proof. We will show that (1) ⇔ (2) and (1) ⇔ (3). Suppose that $L = \sup E$ and let x be a number with $x > L$. That $L \in \sup E$ is clear. We can now assume that $L < +\infty$, for if $L = +\infty$ then there is no such x greater than L. Suppose that $s_n \geq x$ for infinitely many values of n; this forms a subsequence of $\{s_n\}$ consisting of all $s_{n_i} \geq x$. Some subsequence of this subsequence converges to a value y, since $s_{n_i} \geq x$ and $\sup E < +\infty$ implies that $\{s_{n_i}\}$ is bounded by Lemma 15. Then $L \geq y \geq x > L$, which is a contradiction. Conversely, suppose that (2) holds for L and suppose that $L < \sup E$. Then choose x such that $L < x < \sup E$, and there is an integer N such that $n \geq N$ implies $s_n < x$. Every subsequence of $\{s_n\}$ must have a limit no greater than $x < \sup E$ by Theorem 10, and this contradicts the fact that $\sup E$ is the least upper bound. Therefore $L \geq \sup E$, and since $L \in E$, $L = \sup E$. This proves (1) ⇔ (2).

Let $L = \sup E$ so that (2) holds. Let $\varepsilon > 0$ be given. There exists an integer N such that $n \geq N$ implies $s_n < L + \varepsilon/2$. Whenever $n \geq N$, $\sup S_n \leq L + \varepsilon/2$ so that $\sup S_n - L < \varepsilon$. Suppose that $\sup S_n < L$; we can choose x such that $\sup S_n < x < L$. Since every s_k with $k \geq n$ has $s_k < x$, every subsequence of $\{s_n\}$ must have a limit no greater than $x < \sup E$ by Theorem 10. Since L is the least upper bound of E,

$L \le x < L$ which is a contradiction. Therefore $0 \le \sup S_n - L < \varepsilon$, showing that $\lim_{n\to\infty} \sup S_n = L$. This proves (1) $\Leftrightarrow$ (3). □

Theorem 17. *[Exercise 3.5] For any two real sequences $\{a_n\}$ and $\{b_n\}$,*

$$\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided that the sum on the right is not of the form $\infty - \infty$.

Proof. If $\limsup_{n\to\infty} (a_n + b_n) = \pm\infty$ then we are done. Otherwise, let

$$\begin{aligned} L &= \limsup_{n\to\infty} (a_n + b_n), \\ L_1 &= \limsup_{n\to\infty} a_n, \\ L_2 &= \limsup_{n\to\infty} b_n. \end{aligned}$$

There is a subsequence $\{c_{n_i}\}$ of $\{a_n + b_n\}$ that converges to L. For each n_i, $c_{n_i} = a_{n_i} + b_{n_i}$ for some subsequences $\{a_{n_i}\}, \{b_{n_i}\}$ so that $L = a + b$ if we let a be the limit of a_{n_i} and b be the limit of b_{n_i}. Then $L = a + b \le L_1 + L_2$, which proves the result. □

Theorem 18. *[Exercise 3.7] If $a_n \ge 0$ for all n and $\sum a_n$ converges, then $\sum \dfrac{\sqrt{a_n}}{n}$ converges.*

Proof. Let $t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$; clearly $t_n \ge 0$ for all n. Let $b_k = 1/k$, and by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}\right)^2 &\le \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k^2} \\ t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} &\le \sqrt{\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k^2}} \\ &\le \sqrt{ab} \end{aligned}$$

where $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} 1/n^2$. Thus $\{t_n\}$ must be a bounded sequence and hence $\sum \frac{\sqrt{a_n}}{n}$ converges. □

Theorem 19. *[Exercise 3.8] If $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded, then $\sum a_n b_n$ converges.*

Proof. Suppose that $\{b_n\}$ is monotonically increasing and let B be the limit of $\{b_n\}$ so that $b_n \le B$ for every n. Let $C = B \sum a_n - \sum a_n (B - b_n)$. Since $B - b_n \to 0$

and $\{B - b_n\}$ is monotonically decreasing, we can apply Theorem 3.42 to deduce that $\sum a_n (B - b_n)$ converges. Then

$$\begin{aligned} C &= B \sum a_n - \sum a_n (B - b_n) \\ &= \sum a_n b_n \end{aligned}$$

converges. The case for $\{b_n\}$ being monotonically decreasing is similar. □

Theorem 20. *[Exercise 3.10] If $\sum a_n z^n$ is a power series where infinitely many coefficients are distinct from zero, then the radius of convergence is at most 1.*

Proof. Suppose that the radius of convergence $R > 1$, i.e. $\sum a_n \gamma^n$ converges for some $1 < \gamma < R$. By the root test, $\limsup_{n\to\infty} \sqrt[n]{|a_n \gamma^n|} = \limsup_{n\to\infty} \gamma \sqrt[n]{|a_n|} \leq 1$, which means that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L$ where $L < 1$. There exists some subsequence $S = \left\{ \sqrt[n_i]{|a_{n_i}|} \right\}$ that converges to L, and the neighborhood $N_{1-L}(L)$ contains infinitely many points a_k of S with $0 \leq \sqrt[k]{|a_k|} < 1$. But then infinitely many points a_k have $0 \leq |a_k| < 1$, and thus infinitely many points are zero since each a_k is an integer. This is a contradiction, so the radius of convergence must not be greater than 1. □

Theorem 21. *[Exercise 3.11] Suppose that $a_n > 0$, $s_n = a_1 + \cdots + a_n$ and that $\sum a_n$ diverges. Then:*

(1) *The series $\displaystyle\sum \frac{a_n}{1 + a_n}$ diverges.*
(2) *For all $N, k \geq 1$, $\displaystyle\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and $\displaystyle\sum \frac{a_n}{s_n}$ diverges.*
(3) *For all n, $\displaystyle\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and $\displaystyle\sum \frac{a_n}{s_n^2}$ converges.*
(4) *$\displaystyle\sum \frac{a_n}{1 + n a_n}$ sometimes converges and $\displaystyle\sum \frac{a_n}{1 + n^2 a_n}$ always converges.*

Proof. Suppose that $\sum \frac{a_n}{1+a_n}$ converges. Then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$, and $\lim_{n\to\infty} a_n = 0$ (this can be shown using an ε argument). There exists an integer N such that $a_n < 1$ whenever $n \geq N$, and furthermore since $\sum \frac{a_n}{1+a_n}$ converges, for any $\varepsilon > 0$ there exists an integer M such that $\sum_{k=m}^{n} \frac{a_k}{1+a_k} < \varepsilon/2$ whenever $n \geq m \geq M$. Therefore whenever

$n \geq m \geq \max(M, N)$,

$$\begin{aligned} \varepsilon &> 2\sum_{k=m}^{n} \frac{a_k}{1+a_k} \\ &> 2\sum_{k=m}^{n} \frac{a_k}{1+1} \\ &> \sum_{k=m}^{n} a_k \end{aligned}$$

and $\sum a_n$ converges. This shows that $\sum \frac{a_n}{1+a_n}$ diverges if $\sum a_n$ diverges.

For $N, k \geq 1$,

$$\begin{aligned} s_{N+k} - s_N &= a_{N+1} + a_{N+2} + \cdots + a_{N+k} \\ 1 - \frac{s_N}{s_{N+k}} &= \frac{a_{N+1}}{s_{N+k}} + \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &\leq \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+1}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}}. \end{aligned}$$

Suppose that $\sum \frac{a_n}{s_n}$ converges. Then there exists a N such that whenever $n+j \geq n \geq N$,

$$1 - \frac{s_n}{s_{n+j}} \leq \sum_{k=n}^{n+j} \frac{a_n}{s_n} < \frac{1}{2}$$

so that for all j, $2s_n > s_{n+j}$. But $\{s_n\}$ is not bounded since $\sum a_n$ diverges, and there is some j such that $s_{n+j} > 2s_n$. This is a contradiction, so $\sum \frac{a_n}{s_n}$ cannot converge.

For the third inequality,

$$\begin{aligned} 1 &< \frac{s_n}{s_{n-1}} \\ a_n &< \frac{s_n\left(s_n - s_{n-1}\right)}{s_{n-1}} \\ \frac{a_n}{s_n^2} &< \frac{s_n - s_{n-1}}{s_n s_{n-1}} \\ &= \frac{1}{s_{n-1}} - \frac{1}{s_n}. \end{aligned}$$

For any $\varepsilon > 0$, there is some N for which $s_{N-1} > \frac{1}{\varepsilon}$ since $\{s_n\}$ is not bounded. Then for all $n \geq m \geq N$,

$$\begin{aligned}\sum_{k=m}^{n} \frac{a_n}{s_n^2} &< \sum_{k=m}^{n} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right) \\ &< \frac{1}{s_{m-1}} - \frac{1}{s_n} \\ &< \frac{1}{s_{m-1}} - \frac{1}{s_n} \\ &< \varepsilon\end{aligned}$$

since $\{s_n\}$ is monotonically increasing. Hence $\sum \frac{a_n}{s_n^2}$ converges.

The series $\sum \frac{a_n}{1+na_n}$ may or may not converge. If $a_n = 1$ then the series does not converge, but if $a_n = [n = m^2]$ where $[\ldots]$ is the Iverson bracket, then the series converges. The series $\sum \frac{a_n}{1+n^2 a_n}$ always converges since $\frac{a_n}{1+n^2 a_n} = \frac{1}{a_n+n^2} < \sum \frac{1}{n^2}$ and the series on the right hand side converges. □

Theorem 22. *[Exercise 3.12] Suppose that $a_n > 0$ and that $\sum a_n$ converges. Let $r_n = \sum_{m=n}^{\infty} a_m$. Then:*

(1) *If $m < n$ then $\dfrac{a_m}{r_m} + \cdots + \dfrac{a_n}{r_n} > 1 - \dfrac{r_n}{r_m}$, and $\sum \dfrac{a_n}{r_n}$ diverges.*
(2) *For any n, $\dfrac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$, and $\sum \dfrac{a_n}{\sqrt{r_n}}$ converges.*

Proof. If $m < n$ then

$$\begin{aligned}r_m - r_n &< a_m + a_{m+1} + \cdots + a_n \\ 1 - \frac{r_n}{r_m} &< \frac{a_m}{r_m} + \frac{a_{m+1}}{r_m} + \cdots + \frac{a_n}{r_m} \\ &< \frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdots + \frac{a_n}{r_n}.\end{aligned}$$

Suppose that $\sum \frac{a_n}{r_n}$ converges. Then there exists an integer N such that for all $n \geq m \geq N$,

$$1 - \frac{r_n}{r_m} < \sum_{k=m}^{n} \frac{a_k}{r_k} < \frac{1}{2}$$

so that for all $n > m$, $2r_n > r_m$. Since $\sum a_n$ converges, $a_n \to 0$ which means $r_n \to 0$. Hence we can find an integer n such that $r_n < r_m/2$, which is a contradiction. This shows that $\sum \frac{a_n}{r_n}$ does not converge.

To prove the second inequality,

$$\begin{aligned}
4r_n\left(r_n - a_n\right) &< 4r_n^2 - 4a_n r_n + a_n^2 \\
&= \left(2r_n - a_n\right)^2 \\
2\sqrt{r_n}\sqrt{r_n - a_n} &< 2r_n - a_n \\
a_n &< 2\left(r_n - \sqrt{r_n}\sqrt{r_n - a_n}\right) \\
\frac{a_n}{\sqrt{r_n}} &< 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right).
\end{aligned}$$

For any $\varepsilon > 0$, there exists some integer N such that $r_N < \left(\frac{\varepsilon}{2}\right)^2$ since $r_n \to 0$. Then for all $n \geq m \geq N$,

$$\begin{aligned}
\sum_{k=m}^{n} \frac{a_k}{\sqrt{r_k}} &< 2\sum_{k=m}^{n}\left(\sqrt{r_k} - \sqrt{r_{k+1}}\right) \\
&< 2\left(\sqrt{r_m} - \sqrt{r_{n+1}}\right) \\
&< \varepsilon
\end{aligned}$$

since $\{r_n\}$ is monotonically decreasing. Hence $\sum \frac{a_n}{\sqrt{r_n}}$ converges. □

Theorem 23. *[Exercise 3.13] The Cauchy product of two absolutely convergent series converges absolutely.*

Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series; we have $\sum |a_n| \leq M_1$ and $\sum |b_n| \leq M_2$ for some M_1, M_2. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. For all n,

$$\begin{aligned}
\sum_{k=0}^{n} |c_k| &= \sum_{k=0}^{n}\left|\sum_{j=0}^{k} a_j b_{k-j}\right| \\
&\leq \sum_{k=0}^{n}\sum_{j=0}^{k} |a_j|\,|b_{k-j}| \\
&= \sum_{0\leq j\leq k\leq n} |a_j|\,|b_{k-j}| \\
&\leq \sum_{0\leq j,k\leq n} |a_j|\,|b_{n-j}| \\
&= \left(\sum_{j=0}^{n} |a_j|\right)\left(\sum_{k=0}^{n} |b_k|\right) \\
&\leq M_1 M_2
\end{aligned}$$

so that sequence of partial sums of $\sum |c_n|$ is bounded. Therefore $\sum c_n$ converges absolutely. $\square$

Theorem 24. *[Exercise 3.20] Let $\{p_n\}$ be a Cauchy sequence in a metric space X where some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Then the sequence $\{p_n\}$ converges to p.*

Proof. Let $\varepsilon > 0$ be given. There exists some N such that for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon/2$. Also, there exists some K such that for all $k \geq K$, $d(p_{n_k}, p) < \varepsilon/2$. Let j be the smallest integer such that $n_j \geq \max(N, n_K)$. Then for all $n \geq n_j$, $d(p_n, p) \leq d\left(p_n, p_{n_j}\right) + d\left(p_{n_j}, p\right) < \varepsilon$. This shows that $p_n \to p$. $\square$

Theorem 25. *[Exercise 3.21] If $\{E_n\}$ is a sequence of closed, nonempty and bounded sets in a complete metric space X, if $E_n \supseteq E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.*

Proof. Let $\{p_n\}$ be a sequence where each p_i is a point chosen from E_i. Let $\varepsilon > 0$ be given. Since $\operatorname{diam} E_n \to 0$, there exists some N such that $\operatorname{diam} E_n < \varepsilon$ whenever $n \geq N$. Then for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon$ since $p_m, p_n \in E_N$. This shows that $\{p_n\}$ is a Cauchy sequence, and since X is complete, $\{p_n\}$ converges. Suppose that $p \notin E_i$ for some i. Then $p \in E_i^c$ and since E_i^c is open, there exists some neighborhood N of p with radius r such that $N \cap E_i = \emptyset$. In fact, $N \cap E_j = \emptyset$ for every $j \geq i$ since $E_1 \supseteq E_2 \supseteq \cdots$. Since $\{p_n\}$ converges to p, there exists some M such that $d(p_m, p) < r$ whenever $m \geq M$. Let $k = \max(i, M)$ and consider p_k; we have $p_k \in E_k$ but $p_k \in N$ since $k \geq M$, which means that $p_k \notin E_i$ and $p_k \notin E_k$. This is a contradiction, so $p \in E_i$ for all i, i.e. $\bigcap_{n=1}^{\infty} E_n$ is nonempty. Furthermore, since $\operatorname{diam} E_n \to 0$, $\bigcap_{n=1}^{\infty} E_n$ must consist of exactly one point. $\square$

Theorem 26. *[Exercise 3.22, Baire's theorem] If X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X, then $\bigcap_{n=1}^{\infty} G_n$ is not empty.*

Proof. Let g_1 be a point in G_1 and let N_1 be a neighborhood of g_1 wholly contained in G_1. Let E_1 be a neighborhood of g_1 such that $\overline{E_1} \subseteq N_1$. Having constructed $E_1, \ldots, E_n$ such that $E_1 \supseteq \cdots \supseteq E_n$ and $\overline{E_{i+1}} \subset E_i \subseteq G_i$ for each i, let g_n be the center of E_n. Since G_{n+1} is dense in X, E_n contains a point $g_{n+1} \in G_{n+1}$. Let E_{n+1} be a neighborhood of g_{n+1} such that $\overline{E_{n+1}} \subset E_n$. We can continue this process to obtain a sequence $\overline{E_1} \supseteq \overline{E_2} \supseteq \cdots$. By Theorem 25, there is exactly one point $x \in \bigcap_{n=1}^{\infty} \overline{E_n}$. But we have $\overline{E_i} \subseteq G_i$ for each i, which means that $x \in \bigcap_{n=1}^{\infty} G_n$ and therefore $\bigcap_{n=1}^{\infty} G_n$ is not empty. $\square$

Theorem 27. *[Exercise 3.23] Let* $\{p_n\}$ *and* $\{q_n\}$ *be Cauchy sequences in a metric space* X*. Then the sequence* $\{d(p_n, q_n)\}$ *converges.*

Proof. Let $\varepsilon > 0$ be given. There exists, by taking a maximum, an integer N such that for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon/2$ and $d(q_m, q_n) < \varepsilon/2$. Then

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) \end{aligned}$$

and similarly,

$$\begin{aligned} d(p_m, q_m) &\leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m) \\ d(p_m, q_m) - d(p_n, q_n) &\leq d(p_m, p_n) + d(q_n, q_m). \end{aligned}$$

This shows that

$$\begin{aligned} |d(p_n, q_n) - d(p_m, q_m)| &\leq d(p_m, p_n) + d(q_n, q_m) \\ &< \varepsilon \end{aligned}$$

which means that $\{d(p_n, q_n)\}$ converges. □

Theorem 28. *[Exercise 3.24] Let* X *be a metric space.*

(1) *Call two Cauchy sequences* $\{p_n\}, \{q_n\}$ *in* X *equivalent if* $\lim_{n\to\infty} d(p_n, q_n) = 0$*. This is an equivalence relation.*
(2) *Let* X^* *be the set of all equivalence classes obtained by the above equivalence relation. If* $P \in X^*$*,* $Q \in X^*$*,* $\{p_n\} \in P$*,* $\{q_n\} \in Q$*, define* $\triangle(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$*. The number* $\triangle(P, Q)$ *is unchanged if* $\{p_n\}$ *and* $\{q_n\}$ *are replaced by equivalent sequences, and hence that* $\triangle$ *is a distance function in* X^**.*
(3) *The metric space* X^* *is complete.*
(4) *For each* $p \in X$*, there is a Cauchy sequence all of whose terms are* p*; let* P_p *be the element of* X^* *which contains this sequence. Then* $\triangle(P_p, P_q) = d(p, q)$ *for all* $p, q \in X$*.*
(5) *Let* $\varphi : X \to X^*$ *be given by* $p \mapsto P_p$ *where* P_p *is the element of* X^* *which contains a sequence with all terms equal to* p*. Then* $\varphi(X)$ *is dense in* X^**, and if* X *is complete, then* $\varphi(X) = X^*$*.*
(6) *The completion of* $\mathbb{Q}$ *is* $\mathbb{R}$*.*

Proof. It is obvious that that the relation is reflexive and symmetric. Let $\{p_n\}, \{q_n\}, \{r_n\}$ be sequences such that $\lim_{n\to\infty} d(p_n, q_n) = 0$ and $\lim_{n\to\infty} d(q_n, r_n) = 0$. Let $\varepsilon > 0$ be given. There exists, by taking a maximum, an integer N such that for all $n \geq N$, $d(p_n, q_n) < \varepsilon/2$ and $d(q_n, r_n) < \varepsilon/2$. Then

$$\begin{aligned} d(p_n, r_n) &\leq d(p_n, q_n) + d(q_n + r_n) \\ &< \varepsilon, \end{aligned}$$

which shows that $\lim_{n\to\infty} d(p_n, r_n) = 0$. Therefore the relation is transitive.

Let $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$. Let $\{p'_n\} \in P$ and $\{q'_n\} \in Q$ be sequences equivalent to $\{p_n\}$ and $\{q_n\}$ respectively. We must show that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$. Since both limits exist, it suffices to prove that

$$\lim_{n\to\infty} [d(p_n, q_n) - d(p'_n, q'_n)] = 0.$$

From the equivalence of the sequences, we have for any $\varepsilon > 0$ an integer N such that for all $n \geq N$, $d(p_n, p'_n) < \varepsilon/2$ and $d(q_n, q'_n) < \varepsilon/2$. Then

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q_n, q'_n) \\ d(p_n, q_n) - d(p'_n, q'_n) &\leq d(p_n, p'_n) + d(q_n, q'_n) \\ &< \varepsilon \end{aligned}$$

and by symmetry (compare Theorem 27), $|d(p_n, q_n) - d(p'_n, q'_n)| < \varepsilon$. Therefore

$$\lim_{n\to\infty} [d(p_n, q_n) - d(p'_n, q'_n)] = 0,$$

which proves that $\triangle : X^* \times X^* \to \mathbb{R}$ is well-defined. It is simple to verify that $\triangle$ is a metric in X^*.

Let $\{P_n\}$ be a Cauchy sequence in X^*; write $P_n = [\{p_{n,m}\}]$ where $\{p_{n,m}\}$ is a sequence in m. For any $\varepsilon > 0$ there exists some N such that for all $m, n \geq N$, $\triangle(P_m, P_n) < \varepsilon$. **Incomplete.**

Let $p, q \in X$. Then $\triangle(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = d(p, q)$ by definition.

Let $Y = \varphi(X)$ and let $P = [\{p_k\}] \in X^*$ (where $\{p_k\}$ is a representative from the equivalence class), supposing that $P \notin Y$. Let N be a neighborhood of P with radius r. There exists some M such that for all $m, n \geq M$, $d(p_m, p_n) < r$. Let $Q = \varphi(p_M) \in Y$. We want to show that $Q \in N$; we have $d(p_n, p_M) < r$ whenever $n \geq M$, and therefore

$$\triangle(P, Q) = \lim_{n\to\infty} d(p_n, p_M) < r.$$

This proves that $\varphi(X)$ is dense in X^*. **Second part incomplete.** □

Theorem 29. *Let $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ and let a be a limit point of X. If $f(x) \leq g(x)$ for all x in a neighborhood of a, then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x),$$

provided that both limits exist.

Proof. Let N be a neighborhood of a with radius r such that $f(x) \leq g(x)$ for all $x \in N$. Suppose that $\lim_{x\to a}[g(x) - f(x)] = L < 0$. Then there exists a $\delta > 0$ such that $|g(x) - f(x) - L| < -L$ and $g(x) < f(x)$ whenever $0 < |x - a| < \delta$. Choose a point x such that $0 < |x - a| < \min(\delta, r)$; this results in a contradiction. □

Corollary 30. *Let $f, g : [a, \infty) \to \mathbb{R}$. If $f(x) \leq g(x)$ for all $x \geq a$, then*

$$\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x),$$

provided that both limits exist.

Theorem 31. *[Theorem 4.8] A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.*

Proof. Suppose that f is continuous on X. Let V be an open set in Y and let $p \in f^{-1}(V)$. There exists a neighborhood N of $f(p)$ with radius r wholly contained in V. Since f is continuous, there exists a $\delta > 0$ such that $d_Y(f(p), f(x)) < r$ whenever $x \in X$ and $d_X(p, x) < \delta$. Therefore, $N_\delta(p)$ is an open set of X wholly contained in $f^{-1}(V)$. This shows that $f^{-1}(V)$ is an open set. Conversely, suppose that $f^{-1}(V)$ is open in X for every open set V in Y. Let $p \in X$ and let $\varepsilon > 0$ be given. Let V be a neighborhood of $f(p)$ with radius ε so that $f^{-1}(V)$ is open in X. Since $p \in f^{-1}(V)$, there exists a neighborhood N of p with radius δ such that N is wholly contained in $f^{-1}(V)$. Then for all $x \in X$ with $d_X(p, x) < \delta$, we have $d_Y(f(p), f(x)) < \varepsilon$ since $x \in f^{-1}(V)$ and $f(x) \in V$. This shows that f is continuous on X. □

Theorem 32. *[Examples 4.11] The map $x \mapsto |x|$ is continuous.*

Proof. Let $\varepsilon > 0$ be given and let $x, y \in \mathbb{R}^k$ be arbitrary. Whenever $|x - y| < \varepsilon$, we have $||x| - |y|| \leq |x - y| < \varepsilon$, which completes the proof. □

Theorem 33. *[Exercise 4.2] Let f be a continuous map from a metric space X to a metric space Y. Then for every set $E \subseteq X$,*

$$f\left(\overline{E}\right) \subseteq \overline{f(E)}.$$

Furthermore, this inclusion can be proper for certain functions.

Proof. Let $p \in f\left(\overline{E}\right)$; we must show that either $p \in f(E)$ or p is a limit point of $f(E)$. If there is a $x \in E$ with $p = f(x)$, then we are done. Otherwise, $p \notin f(E)$, and we can choose x with $p = f(x)$ such that x is a limit point of E. Let N be a neighborhood of p with radius r. Since f is continuous, there exists a $\delta > 0$ such that for all $y \in N_\delta(x)$ we have $f(y) \in N$. Since x is a limit point of E, there exists a z in $N_\delta(x)$ with $z \in E$ so that $f(z) \in N$. Furthermore, $f(z) \neq p$ since we assumed that $p \notin f(E)$. This shows that p is a limit point of $f(E)$.

The inclusion can be proper, as in the following example. Let $f : (0,1) \to \mathbb{R}$ be defined by $x \mapsto x$; then $f\left(\overline{(0,1)}\right) = (0,1) \neq [0,1] = \overline{f((0,1))}$. □

Theorem 34. *[Exercise 4.3] Let f be a continuous map from a metric space X to $\mathbb{R}$. Let $Z(f)$ be the set of all $p \in X$ such that $f(p) = 0$. Then $Z(f)$ is closed.*

Proof. By definition $Z(f) = f^{-1}(\{0\})$. Since $\{0\}$ is closed and f is continuous, $Z(f)$ must be closed. □

Theorem 35. *[Exercise 4.4] Let f and g be continuous mappings from a metric space X to a metric space Y, and let E be a dense subset of X. Then*

1. *$f(E)$ is dense in $f(X)$, and*
2. *If $g(p) = f(p)$ for all $p \in E$ then $g(p) = f(p)$ for all $p \in X$.*

Proof. We know that $\overline{E} \subseteq X$, and since E is dense in X, $X \subseteq \overline{E}$. By Theorem 33, we have $f\left(\overline{E}\right) = f(X) \subseteq \overline{f(E)}$, which shows that $f(E)$ is dense in $f(X)$.

To prove (2), let $p \in X$. Since E is dense in X, either $p \in E$ or p is a limit point of E. If $p \in E$, then from the assumptions we are done. Otherwise, fix $\varepsilon > 0$. Since f is continuous, there exists a $\delta_1 > 0$ such that for every $x \in N_{\delta_1}(p)$ we have $f(x) \in N_\varepsilon(f(p))$. Similarly, there exists a $\delta_2 > 0$ such that for every $x \in N_{\delta_2}(p)$ we have $g(x) \in N_\varepsilon(g(p))$. Let $\delta = \min(\delta_1, \delta_2)$. Since p is a limit point of E, there exists a point $z \in N_\delta(p)$ with $z \in E$. Then $f(z) \in N_\varepsilon(f(p))$ and $f(z) = g(z) \in N_\varepsilon(g(p))$ so that

$$\begin{aligned} d(f(p), g(p)) &\leq d(f(p), f(z)) + d(f(z), g(p)) \\ &< 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, $f(p) = g(p)$. □

Theorem 36. *[Exercise 4.6] Let E be a subset of $\mathbb{R}$. Define the graph of a function $f : E \to \mathbb{R}$ to be the set $\{(x, f(x)) \mid x \in E\}$. If E is compact, then a function $f : E \to \mathbb{R}$ is continuous if and only if its graph is compact.*

Proof. Let G be the graph of f and let $g : E \to G$ be given by $x \mapsto (x, f(x))$. Clearly, g is a bijection by definition. Suppose that f is continuous. Since $x \mapsto x$ is continuous, by Theorem 4.10 we have that g is continuous. By Theorem 4.14, the image of g is compact, which proves the result. Conversely, suppose that the graph G is compact. Let V be a closed set in $\mathbb{R}$; we want to show that $f^{-1}(V)$ is closed. Let p be a limit point of $f^{-1}(V)$. By Theorem 3.2, there exists a sequence $\{p_n\}$ in $f^{-1}(V)$ that converges to p. Consider the sequence $\{(p_n, f(p_n))\}$; since G is compact, some subsequence $\{(p_{n_i}, f(p_{n_i}))\}$ converges to some $(p, y) \in G$, and by definition, $y = f(p)$. Now $\{f(p_{n_i})\}$ is a sequence in V, and since V is closed and the sequence converges to $f(p)$, we have $f(p) \in V$. Therefore $p \in f^{-1}(V)$, which shows that $f^{-1}(V)$ is closed. □

Theorem 37. *[Exercise 4.8] Let E be a bounded set in $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Then f is bounded on E. If E is not bounded, then the conclusion does not necessarily hold.*

Proof. We can choose M, N so that $M < x < N$ for all $x \in E$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Choose n so that $N - M + \delta > (n+1)\delta \geq N - M$. For every $x \in E$, there is an integer k with $0 \leq k \leq n$ such that $|M + k\delta - x| < \delta$. Then $|f(M + k\delta) - f(x)| < 1$ which means $|f(x)| < 1 + |f(M + k\delta)|$. Now take

$$P = \min_{0 \leq k \leq n} |f(M + k\delta)|$$

where $k = 0, 1, \ldots, n$; we have $|f(x)| < 1 + P$ for all $x \in E$ and hence f is bounded on E.

To show that E must be bounded for the conclusion to hold, choose $f(x) = x$, which is uniformly continuous, and $E = \mathbb{R}$. □

Theorem 38. *[Exercise 4.9] Let $f : X \to Y$. Then the following statements are equivalent:*

(1) *f is uniformly continuous.*
(2) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that* $\operatorname{diam} f(E) < \varepsilon$ *whenever $E \subseteq X$ and* $\operatorname{diam} E < \delta$.

Proof. Obvious. □

Theorem 39. *Let X and Y be metric spaces. Let $f : X \to Y$ be a continuous function. If $\{s_n\}$ is a sequence in X that converges to s, then $\{f(s_n)\}$ converges to $f(s)$.*

Proof. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that $d(f(s), f(x)) < \varepsilon$ whenever $d(s, x) < \delta$. Since $s_n \to s$, there exists a N such that for all $n \geq N$ we have $d(s, s_n) < \delta$. Then $d(f(s), f(s_n)) < \varepsilon$ whenever $n \geq N$, which completes the proof. □

Theorem 40. *Let X, Y, Z be metric spaces. Let $f : X \to Y$ be a function with $\lim_{x\to a} f(x) = b$ and let $g : Y \to Z$ be continuous at b. Then $\lim_{x\to a} g(f(x)) = g(b)$.*

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $d_Z(g(x), g(b)) < \varepsilon$ whenever $d_Y(x, b) < \delta$, and choose $\gamma > 0$ such that $d_Y(f(x), b) < \delta$ whenever $0 < d_X(x, a) < \gamma$. Then $d_Z(g(f(x)), g(b)) < \varepsilon$ whenever $0 < d_X(x, a) < \gamma$. □

Theorem 41. *Let X and Y be metric spaces. Let $f : X \to Y$ be a function with*

$$\lim_{x\to a} f(x) = L.$$

If F is any neighborhood of a and $g : E \to F$ is a continuous bijection where $g^{-1}(a)$ is a limit point of E, then

$$\lim_{x\to g^{-1}(a)} f(g(x)) = L.$$

Proof. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x), L) < \varepsilon$ whenever $0 < d(x, a) < \delta$. Since g is continuous on E, there exists a $\gamma > 0$ such that $d(g(x), a) < \delta$ whenever $d(x, g^{-1}(a)) < \gamma$. Then for all x with $0 < d(x, g^{-1}(a)) < \gamma$ we have $0 < d(g(x), a) < \delta$, noting that $d(g(x), a) = 0$ if and only if $d(x, g^{-1}(a)) = 0$, since g is a bijection. Therefore, $d(f(g(x)), L) < \varepsilon$, which completes the proof. □

Theorem 42. *[Exercise 4.10] Let X be a compact metric space and let Y be a metric space. If $f : X \to Y$ is a continuous function, then f is also uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then there exists a $\varepsilon > 0$ such that for every $\delta > 0$ we have some $E \subseteq X$ with $\operatorname{diam} E < \delta$ such that $\operatorname{diam} f(E) \geq \varepsilon > \gamma$, where $\gamma = \varepsilon/2$. Let $\delta_n = 1/n$; for each n we have points $p_n, q_n \in X$ such that $d_X(p_n, q_n) < \delta_n$ and $d_Y(f(p_n), f(q_n)) > \gamma$. Since X is compact, some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. By Theorem 39, the sequence $\{f(p_{n_i})\}$ converges to $f(p)$. Similarly we have $q_{n_i} \to p$ and $f(q_{n_i}) \to f(p)$ upon application of Theorem 13 and Theorem 39. Now there exist integers M, N such that $d_Y(f(p), f(p_{n_i})) < \gamma/2$ whenever $n_i \geq M$, and $d_Y(f(p), f(q_{n_i})) < \gamma/2$ whenever $n_i \geq N$. Taking n_i to be an integer with $n_i \geq \max(M, N)$, we find that

$$\begin{aligned} d_Y(f(p_{n_i}), f(q_{n_i})) &\leq d_Y(f(p_{n_i}), f(p)) + d_Y(f(p), f(q_{n_i})) \\ &< \gamma, \end{aligned}$$

which is a contradiction. □

Theorem 43. *[Exercise 4.11] Let X and Y be metric spaces. If $f : X \to Y$ is a uniformly continuous function, then $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists a $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Since $\{x_n\}$ is a Cauchy sequence, there exists a N such that $d(x_i, x_j) < \delta$ whenever $i, j \geq N$. Then for all $i, j \geq N$ we have $d(f(x_i), f(x_j)) < \varepsilon$, which completes the proof. □

Theorem 44. *[Exercise 4.12] Let X, Y, Z be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions, then $h = g \circ f$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. There exists a $\delta_1 > 0$ such that $d_Z(g(x), g(y)) < \varepsilon$ whenever $d_Y(x, y) < \delta_1$. There also exists a $\delta_2 > 0$ such that $d_Y(f(x), f(y)) < \delta_1$ whenever $d_X(x, y) < \delta_2$. Then for all x, y with $d_X(x, y) < \delta_2$ we have

$$d_Y(f(x), f(y)) < \delta_1$$

and

$$d_Z(g(f(x)), g(f(y))) = d_Z(h(x), h(y)) < \varepsilon.$$

□

Lemma 45. *Let X, Y be metric spaces and let $f : X \to Y$ be a uniformly continuous function. Let $\{x_n\}, \{y_n\}$ be sequences in X that both converge to $x \in X$. If $f(x_n) \to y$ and $f(y_n) \to z$, then $y = z$.*

Proof. Fix $\varepsilon > 0$. Since f is uniformly continuous, there is some $\delta > 0$ such that $d(f(a), f(b)) < \varepsilon/3$ whenever $d(a, b) < \delta$. For some N we have $d(x, x_n) < \delta/2$ and $d(x, y_n) < \delta/2$ whenever $n \geq N$, so that

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \\ &< \delta \end{aligned}$$

and therefore $d(f(x_n), f(y_n)) < \varepsilon/3$ whenever $n \geq N$. Furthermore, there exist integers N_1, N_2 such that $d(y, f(x_n)) < \varepsilon/3$ whenever $n \geq N_1$ and $d(z, f(y_n)) < \varepsilon/3$ whenever $n \geq N_2$. Setting $n = \max\{N, N_1, N_2\}$, we have

$$\begin{aligned} d(y, z) &\leq d(y, f(x_n)) + d(f(x_n), z) \\ &\leq d(y, f(x_n)) + d(f(x_n), f(y_n)) + d(f(y_n), z) \\ &< \varepsilon. \end{aligned}$$

Since ε was arbitrary, $y = z$. □

Theorem 46. *[Exercise 4.13] Let E be a dense subset of a metric space X, and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Then f has a continuous extension from E to X.*

Proof. We will define $g : X \to \mathbb{R}$ as follows. Let $x \in X$. Since E is dense in X, there exists a sequence $\{x_n\}$ in E that converges to x. Then $\{x_n\}$ is a Cauchy sequence, and by Theorem 43, $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. By Theorem 3.11, there exists some $y \in \mathbb{R}$ such that $f(x_n) \to y$. We may then define $g(x) = y$ in this manner, noting that it is well-defined by Lemma 45. Now we will prove that g is continuous. Let $\varepsilon > 0$ and $x \in X$ be given. Since f is uniformly continuous, there exists a $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon/3$ whenever $d(x, x') < \delta$. As in our construction of g, there exists a sequence $\{x_n\}$ in E that converges to x, while $f(x_n) \to y$ for some $y \in \mathbb{R}$. Then there exists a M such that for every $n \geq M$ we have $d(y, f(x_n)) < \varepsilon/3$. Now let $x' \in X$ with $d(x, x') < \delta$. There exists a sequence $\{x'_n\}$ in E that converges to x', while $f(x'_n) \to y'$ for some $y' \in \mathbb{R}$. Then there exists a N such that for every $n \geq N$ we have $d(y', f(x'_n)) < \varepsilon/3$. Now take $n = \max(M, N)$, and then

$$\begin{aligned} d(f(x), f(x')) &= d(y, y') \\ &\leq d(y, f(x_n)) + d(f(x_n), f(x'_n)) + d(f(x'_n), y') \\ &< \varepsilon. \end{aligned}$$

This shows that g is a continuous extension of f from E to X. Note that we may replace the range of f with any complete metric space. ☐

Theorem 47. *[Exercise 4.14] Let $I = [0, 1]$ be the closed unit interval. If $f : I \to I$ is a continuous function, then $f(x) = x$ for at least one $x \in I$.*

Proof. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - x$. If $f(0) = 0$ or $f(1) = 1$ then we are done. Therefore, we may assume that $f(0) > 0$ and $f(1) < 1$. We have $g(0) = f(0) > 0$ while $g(1) = f(1) - 1 < 0$. By the intermediate value theorem, there exists a $x \in (0, 1)$ such that $g(x) = 0$, i.e. $f(x) = x$. ☐

Lemma 48. *If a function $f : \mathbb{R} \to \mathbb{R}$ is not monotonic, then there exist points p_1, p_2, p_3 such that $p_1 < p_2 < p_3$, and either $f(p_1), f(p_3) < f(p_2)$ or $f(p_1), f(p_3) > f(p_2)$.*

Proof. If f is not monotonic, then there exist points x_1, y_1, x_2, y_2 such that $x_1 < y_1$, $f(x_1) < f(y_1)$, $x_2 < y_2$, $f(x_2) > f(y_2)$. We can construct a list of all possible orderings to prove the result. ☐

Theorem 49. *[Exercise 4.15] Every continuous open map from $\mathbb{R}$ to $\mathbb{R}$ is monotonic.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous open map. Suppose that f is not monotonic. By Lemma 48, there exist points p_1, p_2, p_3 such that $p_1 < p_2 < p_3$, and either $f(p_1), f(p_3) < f(p_2)$ or $f(p_1), f(p_3) > f(p_2)$. Assume without loss of generality that $f(p_1), f(p_3) < f(p_2)$, and let $M = \sup f([p_1, p_3])$. Then by Theorem 4.16 there exists a point $x \in [p_1, p_3]$ such that $f(x) = M$. Let $V = (p_1, p_3)$; then $x \in V$ since $f(p_1), f(p_3) < f(p_2) \leq M$. Since f is an open map, $f(V)$ is open, and there exists a neighborhood

N of $f(x)$ with radius r such that $N \subseteq f(V)$. Then $f(x) + r/2 \in f(V)$, which means that $f(x') > M$ for some $x' \in V$. This is a contradiction, so f must be monotonic. □

Theorem 50. *[Exercise 4.17] The set of points at which a function $f : (a, b) \to \mathbb{R}$ has a simple discontinuity is at most countable.*

Proof. Let E the set of all $x \in (a, b)$ such that $f(x-) < f(x+)$. For each $x \in E$, associate with x a triple (p, q, r):

(1) Choose $p \in \mathbb{Q}$ so that $f(x-) < p < f(x+)$.
(2) There exists a $\delta > 0$ such that $|f(t) - f(x-)| < p - f(x-)$ whenever $x - \delta < t < x$. Choose $q \in \mathbb{Q}$ so that $x - \delta < q < x$. Then whenever $a < q < t < x$ we have $f(t) < p$.
(3) There exists a $\delta > 0$ such that $|f(x+) - f(t)| < f(x+) - p$ whenever $x < t < x + \delta$. Choose $r \in \mathbb{Q}$ so that $x < r < x + \delta$. Then whenever $x < t < r < b$ we have $f(t) > p$.

Now we must prove that each triple is associated with at most one $x \in E$. Let $x, y \in E$ such that x, y are both associated with the triple (p, q, r). We obtain four inequalities:

$$
\begin{aligned}
f(t) &< p \text{ whenever } a < q < t < x, \\
f(t) &> p \text{ whenever } x < t < r < b, \\
f(t) &< p \text{ whenever } a < q < t < y, \\
f(t) &> p \text{ whenever } y < t < r < b.
\end{aligned}
$$

Suppose that $x < y$. We can choose u with $x < u < y$. Since $x < u < r$, we have $f(u) > p$, and since $q < u < y$, we have $f(u) < p$, which is a contradiction. Similarly, we obtain a contradiction if $x > y$. Therefore $x = y$. Let F be the set of all $x \in (a, b)$ such that $f(x-) > f(x+)$; we can again associate with $x \in F$ a triple (p, q, r). For the last kind of simple discontinuity, let G be the set of all $x \in (a, b)$ such that $f(x-) = f(x+)$ but $f(x) \neq f(x-), f(x+)$. For each $x \in G$, associate with x a tuple (q, r) where q, r are defined in a similar way to the triples (p, q, r) associated with E. The sets E, F, G are all countable, so the result follows. □

Theorem 51. *[Exercise 4.19] Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the following property: if $f(a) < c < f(b)$, then $f(x) = c$ for some $x \in (a, b)$. Also, for every $r \in \mathbb{Q}$, the set of all x with $f(x) = r$ is closed. Then f is continuous.*

Proof. Suppose that f is not continuous. Then there exist $\varepsilon > 0$ and $x \in \mathbb{R}$ such that for all $\delta > 0$ we have $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$ for some y. Put $\delta_n = 1/n$ to form a sequence $x_n \to x$ while $|f(x) - f(x_n)| \geq \varepsilon$ for all n. Either x_n has a infinite number of points with $f(x) < f(x_n)$, or an infinite number of points with $f(x_n) < f(x)$. Assume without loss of generality that the former holds, so that there exists a subsequence $x_{n_i} \to x$ with $f(x) + \varepsilon \leq f(x_n)$ for all n. Let r be some rational number with $f(x) < r < f(x) + \varepsilon$. For all n we have $f(x) < r < f(x_n)$; by the given property of f, there exists a $t_n \in (x, x_n)$ with $f(t_n) = r$, and with the sequence t_n converging to x since $x_n \to x$. Let E be the set of all a with $f(a) = r$. Since $t_n \to x$ and $f(t_n) = r$, we have that x is a limit point of E. But $f(x) < r$, so E is not closed. This is a contradiction, and therefore f must be continuous. □

Theorem 52. *[Exercise 4.20] If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by*

$$p_E(x) = \inf_{z \in E} d(x, z).$$

Then:

(1) $p_E(x) = 0$ *if and only if* $x \in \overline{E}$.
(2) p_E *is a uniformly continuous function on* X.

Proof. Suppose that $p_E(x) = 0$ and $x \notin E$. Let N be a neighborhood of x with radius r; by definition of the infimum, N contains a point $z \in E$ with $d(x, z) < r$ (and $z \neq x$). Hence x is a limit point of E. Conversely, suppose that $p_E(x) = L$ with $L > 0$. Clearly $x \notin E$ since $d(x, x) = 0$. Also, x is not a limit point of E since the neighborhood $N_L(x)$ contains no points in E. Therefore $x \notin \overline{E}$.

Fix $x, y \in X$. Then for all $z \in E$ we have

$$p_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Therefore $d(y, z) \geq p_E(x) - d(x, y)$ for all z, which means that $p_E(y) \geq p_E(x) - d(x, y)$. Similarly, $p_E(x) \geq p_E(y) - d(x, y)$, and thus

$$|p_E(x) - p_E(y)| \leq d(x, y).$$

Whenever $d(x, y) < \varepsilon$ we have $|p_E(x) - p_E(y)| < \varepsilon$, which shows that p_E is uniformly continuous. □

Theorem 53. *[Exercise 4.21] Let K and F be disjoint sets in a metric space X, with K compact and F closed. Then there exists a $\delta > 0$ such that $d(p, q) > \delta$ for all $p \in K$ and $q \in F$.*

Proof. Consider the map $p_F : K \to \mathbb{R}$ defined in Theorem 52. Suppose that $p_F(x) = 0$ for some $x \in K$. Then by Theorem 52, $x \in \overline{F} = F$, which is a contradiction. Therefore $p_F(x) > 0$ for all $x \in K$. Let $D = p_F(K)$; since K is compact, D is compact, and additionally D is closed by the Heine-Borel theorem. Since $0 \in D^c$ and D^c is open, there exists a neighborhood N of 0 with radius $r > 0$ such that $N \subseteq D^c$. Therefore, $p_F(x) \geq r$ for all $x \in K$, and the result follows. □

Theorem 54. *[Exercise 4.23] If $f:(a,b)\to\mathbb{R}$ is a convex function and $a<s<t<u<b$, then*

$$\frac{f(t)-f(s)}{t-s}\le\frac{f(u)-f(s)}{u-s}\le\frac{f(u)-f(t)}{u-t} \tag{*}$$

and f is continuous. Additionally, every increasing convex function of a convex function is convex.

Proof. We have

$$\begin{aligned}t&=\frac{t-s}{u-s}u+\left(1-\frac{t-s}{u-s}\right)s\\&=\frac{u-t}{u-s}s+\left(1-\frac{u-t}{u-s}\right)u.\end{aligned}$$

Then

$$\begin{aligned}f(t)&\le\frac{t-s}{u-s}f(u)+\left(1-\frac{t-s}{u-s}\right)f(s)\\\frac{f(t)-f(s)}{t-s}&\le\frac{f(u)-f(s)}{u-s}\end{aligned}$$

and

$$\begin{aligned}f(t)&\le\frac{u-t}{u-s}f(s)+\left(1-\frac{u-t}{u-s}\right)f(u)\\\frac{f(u)-f(s)}{u-s}&\le\frac{f(u)-f(t)}{u-t}.\end{aligned}$$

Let $x\in(a,b)$ and choose δ so that $[x-\delta,x+\delta]\in(a,b)$. Let $y\in(x-\delta,x+\delta)\setminus\{x\}$. We want to show that the following inequality holds:

$$\frac{f(x)-f(x-\delta)}{\delta}\le\frac{f(x)-f(y)}{x-y}\le\frac{f(x+\delta)-f(x)}{\delta}.$$

If $y<x$, then applying (*) on $x-\delta<y<x$ and $y<x<x+\delta$ produces the result. Similarly, if $y>x$ then applying (*) on $x-\delta<x<y$ and $x<y<x+\delta$ produces the result. Then for all $y\in(x-\delta,x+\delta)$, $|f(x)-f(y)|\le C|x-y|$ for some positive constant C. This proves that f is continuous.

Let $g:(c,d)\to\mathbb{R}$ be an increasing convex function where the range of f is a subset of (c,d). Then for all $x,y\in(a,b)$ and $\lambda\in(0,1)$,

$$\begin{aligned}f(\lambda x+(1-\lambda)y)&\le\lambda f(x)+(1-\lambda)f(y)\\g(f(\lambda x+(1-\lambda)y))&\le g(\lambda f(x)+(1-\lambda)f(y))\\&\le\lambda g(f(x))+(1-\lambda)g(f(y)),\end{aligned}$$

which shows that $g \circ f$ is convex. $\square$

Definition 55. Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is **midpoint convex** if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in I$. A **binary sequence** is a sequence $\{b_n\}$ where every b_n is either 0 or 1.

Lemma 56. *Let $f : [0,1] \to \mathbb{R}$ be a midpoint convex function and let $\{b_n\}$ be a binary sequence. Let $\lambda_n = \sum_{k=1}^{n} b_k 2^{-k}$. Then*

$$f(\lambda_n) \leq \lambda_n f(1) + (1-\lambda_n) f(0).$$

Proof. We first use induction on n to prove that

$$f\left(\sum_{k=1}^{n} b_k 2^{-k}\right) \leq \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0) 2^{-n}$$

for any binary sequence $\{b_n\}$. If $n = 1$ and $b_1 \in \{0, 1\}$, then

$$f\left(\frac{b_1}{2}\right) = f\left(\frac{0+b_1}{2}\right) \leq \frac{1}{2} f(b_1) + \frac{1}{2} f(0)$$

since f is midpoint convex. Otherwise, assuming the statement for $n-1$, we have for any binary sequence $\{b_n\}$,

$$\begin{aligned}
f\left(\sum_{k=1}^{n} b_k 2^{-k}\right) &= f\left(\frac{1}{2}\left[b_1 + \sum_{k=2}^{n} b_k 2^{-k+1}\right]\right) \\
&\leq \frac{1}{2} f(b_1) + \frac{1}{2} f\left(\sum_{k=2}^{n} b_k 2^{-k+1}\right) \\
&\leq \frac{1}{2} f(b_1) + \frac{1}{2} \sum_{k=2}^{n} f(b_k) 2^{-k+1} + f(0) 2^{-n} \\
&= \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0) 2^{-n},
\end{aligned}$$

which proves the statement for all n. We now compute

$$\begin{aligned} 1-\lambda_n &= \sum_{k=1}^{\infty} 2^{-k} - \sum_{k=1}^{n} b_k 2^{-k} \\ &= \sum_{k=1}^{n} (1-b_k)2^{-k} + \sum_{k=n+1}^{\infty} 2^{-k} \\ &= \sum_{k=1}^{n} (1-b_k)2^{-k} + 2^{-n} \end{aligned}$$

so that

$$\begin{aligned} \lambda_n f(1) + (1-\lambda_n) f(0) &= \sum_{k=1}^{n} f(1) b_k 2^{-k} + \sum_{k=1}^{n} f(0)(1-b_k)2^{-k} + f(0)2^{-n} \\ &= \sum_{k=1}^{n} f(b_k) 2^{-k} + f(0)2^{-n} \end{aligned}$$

since b_k is always 0 or 1, and $f(1)b_k + f(0)(1-b_k)$ is always equal to $f(b_k)$. This proves the result. □

Theorem 57. *[Exercise 4.24] Let $f : (a,b) \to \mathbb{R}$ be a continuous, midpoint convex function. Then f is convex.*

Proof. We first prove a smaller result for any continuous, midpoint convex function $g : [0,1] \to \mathbb{R}$. Let $\lambda \in (0,1)$ and let $\{b_n\}$ be a binary expansion of λ so that if $\lambda_n = \sum_{k=1}^{n} b_k 2^{-k}$, then $\lambda_n \to \lambda$. By Lemma 56, we have $g(\lambda_n) \leq \lambda_n g(1) + (1-\lambda_n) g(0)$, and by Theorem 39, $g(\lambda_n) \to g(\lambda)$. Therefore by Theorem 10,

$$(*) \qquad g(\lambda) \leq \lambda g(1) + (1-\lambda) g(0).$$

For the general case, let $x, y \in (a,b)$ and let $\lambda \in (0,1)$. If $x = y$, then we are done. Otherwise, assume without loss of generality that $x < y$. Define $g : [0,1] \to \mathbb{R}$ by $g(\lambda) = f(\lambda y + (1-\lambda)x)$. For any $\lambda_1, \lambda_2 \in [0,1]$, we have

$$\begin{aligned} g\left(\frac{\lambda_1+\lambda_2}{2}\right) &= f\left(x + \frac{\lambda_1+\lambda_2}{2}(y-x)\right) \\ &= f\left(\frac{[\lambda_1 y + (1-\lambda_2)x] + [\lambda_2 y + (1-\lambda_2)x]}{2}\right) \\ &\leq \frac{g(\lambda_1)+g(\lambda_2)}{2}, \end{aligned}$$

which shows that g is midpoint convex. By (*),

$$g(\lambda) \leq \lambda g(1) + (1-\lambda) g(0)$$
$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x)$$

for all $\lambda \in (0,1)$. This proves that f is convex. □

Theorem 58. *[Exercise 4.26] Let X, Y, Z be metric spaces with Y compact. Let $f : X \to Y$ such that $f(X) \subseteq Y$, and let $g : Y \to Z$ be a continuous, injective function. Let $h : X \to Z$ be defined by $h(x) = g(f(x))$. Then:*

(1) *If h is uniformly continuous, then f is uniformly continuous.*
(2) *If h is continuous, then f is continuous.*

Proof. Suppose that h is uniformly continuous. Since g is continuous and Y is compact, $g(Y)$ is compact. Since g is injective, $f(x) = g^{-1}(h(x))$, and $g^{-1} : g(Y) \to Y$ is continuous by Theorem 4.17. But $g(Y)$ is compact, so by Theorem 4.19, g^{-1} is uniformly continuous. Applying Theorem 44 proves that f is uniformly continuous.

Suppose that h is continuous. Again, $f = g^{-1} \circ h$, and g^{-1} is continuous by Theorem 4.17. Applying Theorem 4.7 proves that f is continuous. □

Chapter 5. Differentiation

Lemma 59. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function differentiable at x. Then there exists a function $\phi : I \to \mathbb{R}$ such that*

$$f(t) - f(x) = (t-x)[f'(x) + \phi(t)]$$

for all $t \in I$ and

$$\lim_{t \to x} \phi(t) = \phi(0) = 0.$$

Proof. Define

$$\phi(t) = \begin{cases} 0 & \text{if } t = x, \\ \frac{f(t)-f(x)}{t-x} - f'(x) & \text{otherwise.} \end{cases}$$

This function clearly satisfies the desired properties. □

Theorem 60. *Let I_1, I_2 be intervals. Let $f : I_1 \to \mathbb{R}$ be a continuous function and let $g : I_2 \to \mathbb{R}$ be a function where I_2 contains the range of f. Define $h : I_1 \to \mathbb{R}$ by $h(x) = g(f(x))$. If f is differentiable at some point $x \in I_1$ and g is differentiable at $f(x)$, then $h'(x) = g'(f(x))f'(x)$.*

Proof. Let $y = f(x)$ for convenience. By Lemma 59, there exist functions ϕ_1, ϕ_2 with

$$\lim_{t\to x} \phi_1(t) = \lim_{s\to y} \phi_2(s) = 0$$

such that

$$f(t) - f(x) = (t - x)[f'(x) + \phi_1(t)],$$
$$g(s) - g(y) = (s - y)[g'(y) + \phi_2(s)],$$

whenever $t \in I_1$ and $s \in I_2$. In particular, by setting $s = f(t)$ we have for all $t \in I_1$,

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= (f(t) - f(x))[g'(f(x)) + \phi_2(f(t))] \\ &= (t - x)[f'(x) + \phi_1(t)][g'(f(x)) + \phi_2(f(t))], \end{aligned}$$

so that

$$(*) \qquad \frac{h(t) - h(x)}{t - x} = [f'(x) + \phi_1(t)][g'(f(x)) + \phi_2(f(t))]$$

if $t \neq x$. By Theorem 40,

$$\lim_{t\to x} \phi_2(f(t)) = \phi_2(f(x)) = 0$$

since f is continuous at x and ϕ_2 is continuous at $f(x)$, so taking $t \to x$ in (*) completes the proof. □

Theorem 61. *[Exercise 5.1] Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Then f is constant.

Proof. The condition on f is that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

for all $x, y \in \mathbb{R}$. Then $f'(x) = 0$ for all x, and by the mean value theorem, f is constant. □

Theorem 62. *[Exercise 5.2] Let $f : (a, b) \to \mathbb{R}$ with $f'(x) > 0$ for all $x \in (a, b)$. Then:*

(1) *f is strictly increasing in (a, b), and*
(2) *If g is the inverse function of f, then g is differentiable and*

$$g'(f(x)) = \frac{1}{f'(x)}$$

for all $x \in (a, b)$.

Proof. Let $x, y \in (a, b)$ with $x < y$. By the mean value theorem, there exists a $c \in (x, y)$ such that $f(y) - f(x) = (y - x)f'(c) > 0$, and therefore $f(x) < f(y)$. This shows that f is strictly increasing in (a, b). Let $x \in (a, b)$; we want to show that g is differentiable at $f(x)$. Since f is differentiable at x, we have

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x).$$

By Theorem 4.4, since $f'(x) > 0$,

$$\lim_{t \to x} \frac{t - x}{f(t) - f(x)} = \frac{1}{f'(x)}.$$

By Theorem 41 applied with g, we have

$$\lim_{t \to f(x)} \frac{g(t) - g(f(x))}{t - f(x)} = \frac{1}{f'(x)}$$

and therefore $g'(f(x)) = 1/f'(x)$. □

Theorem 63. *[Exercise 5.3] Let $g : \mathbb{R} \to \mathbb{R}$ with a bounded derivative $|g'| \leq M$. Fix $\varepsilon > 0$ and let $f(x) = x + \varepsilon g(x)$. Then f is injective if ε is small enough.*

Proof. Take $\varepsilon < 1/M$. Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$, i.e. $x + \varepsilon g(x) = y + \varepsilon g(y)$, so that

$$\left| \frac{g(x) - g(y)}{x - y} \right| = \frac{1}{\varepsilon}.$$

Suppose that $x \neq y$; then by the mean value theorem, there exists a $z \in (x, y)$ such that

$$|g'(z)| = \left| \frac{g(x) - g(y)}{x - y} \right| = \frac{1}{\varepsilon} \leq M.$$

This is a contradiction since $1/\varepsilon > M$, so $x = y$ whenever $f(x) = f(y)$. □

Theorem 64. *[Exercise 5.4] If $C_0, \ldots, C_n$ are real constants such that*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

then the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let

$$f(x) = C_0 x + \frac{C_1}{2} x + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1}$$

so that $f(0) = f(1) = 0$. By the mean value theorem, there exists a $x \in (0, 1)$ such that

$$f'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0.$$

□

Theorem 65. *[Exercise 5.5] Let f be defined and differentiable for every $x > 0$, with $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x+1) - f(x)$. Then $g(x) \to 0$ as $x \to +\infty$.*

Proof. For every $\varepsilon > 0$, there exists a $M > 0$ such that $|f'(x)| < \varepsilon$ whenever $x > M$. Then for all $x > M$, applying the mean value theorem to f gives a $c \in (x, x+1)$ such that $f(x+1) - f(x) = f'(c)$. Since $c > M$, we have $|f(x+1) - f(x)| = |f'(c)| < \varepsilon$, which proves that $g(x) \to 0$ as $x \to +\infty$. □

Theorem 66. *[Exercise 5.6] Let f be a real function. Suppose that*

(1) *f is continuous for $x \geq 0$,*
(2) *$f'(x)$ exists for $x > 0$,*
(3) *$f(0) = 0$,*
(4) *f' is monotonically increasing.*

Let

$$g(x) = \frac{f(x)}{x}$$

be defined for all $x > 0$. Then g is monotonically increasing.

Proof. The derivative of g is given by

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

so we want to prove that $xf'(x) - f(x) > 0$ for all $x > 0$. For all $x > 0$, by the mean value theorem, there exists a $c \in (0, x)$ such that

$$\frac{f(x)}{x} = f'(c) < f'(x)$$

since $c < x$ and f' is monotonically increasing. This proves the result. □

Theorem 67. *[Exercise 5.7] Suppose that $f'(x)$ and $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Then*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Proof. Since $f'(x)$ and $g'(x)$ exist, we have

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=\lim_{t\to x}\frac{f(t)}{t-x}=f'(x),$$
$$\lim_{t\to x}\frac{g(t)-g(x)}{t-x}=\lim_{t\to x}\frac{g(t)}{t-x}=g'(x).$$

Since $g'(x)\neq 0$, by Theorem 4.4 the result follows. □

Theorem 68. *[Exercise 5.8] Suppose that f' is continuous on $[a,b]$ and $\varepsilon>0$. Then there exists a $\delta>0$ such that*

$$\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|<\varepsilon$$

whenever $0<|t-x|<\delta$ and $t,x\in[a,b]$.

Proof. By Theorem 4.19, f' is uniformly continuous since $[a,b]$ is compact. There exists a $\delta>0$ such that $|f'(t)-f'(x)|<\varepsilon$ whenever $|t-x|<\delta$. Then for all $t,x\in[a,b]$ with $0<|t-x|<\delta$, by the mean value theorem, there exists a $u\in(t,x)$ such that

$$\left|\frac{f(t)-f(x)}{t-x}-f'(u)\right|=0,$$

and

$$\begin{aligned}\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right| &\leq \left|\frac{f(t)-f(x)}{t-x}-f'(u)\right|+|f'(u)-f'(c)|\\ &<\varepsilon.\end{aligned}$$

□

Theorem 69. *[Exercise 5.9] Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f'(x)$ exists for all $x\neq 0$ and $f'(x)\to 3$ as $x\to 0$. Then $f'(0)$ exists.*

Proof. For every $\varepsilon>0$ there exists a $\delta>0$ such that $|f'(x)-3|<\varepsilon$ whenever $0<|x|<\delta$. For all x with $0<|x|<\delta$, by the mean value theorem, there exists a $c\in(0,x)$ such that

$$\frac{f(x)-f(0)}{x}=f'(c)$$
$$\left|\frac{f(x)-f(0)}{x}-3\right|=|f'(c)-3|<\varepsilon.$$

□

Theorem 70. *[Exercise 5.11] Suppose that f is defined in a neighborhood of x, and suppose that $f''(x)$ exists. Then*

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x).$$

Proof. Since $f''(x)$ exists, we have

$$\begin{aligned} f''(x) &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} \\ &= \lim_{h\to 0}\frac{f'(x)-f'(x-h)}{h} \end{aligned}$$

where the second limit is obtained by applying Theorem 41 with the bijection $h\mapsto -h$. Adding the two limits gives

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}.$$

As $h\to 0$ we have $f(x+h)+f(x-h)-2f(x)\to 0$ and $h^2\to 0$, so by Theorem 5.13,

$$\begin{aligned} \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \\ &= f''(x). \end{aligned}$$

□

Theorem 71. *[Exercise 5.14] Let $f:(a,b)\to\mathbb{R}$ be a differentiable function. Then f is convex if and only if f' is monotonically increasing. If $f''(x)$ exists for all $x\in(a,b)$, then f is convex if and only if $f''(x)\geq 0$ for all $x\in(a,b)$.*

Proof. Suppose that f is convex. Let $x,y\in(a,b)$ with $x<y$. Since f is convex, every $t\in(x,y)$ has

$$\frac{f(t)-f(x)}{t-x}\leq\frac{f(y)-f(x)}{y-x}\leq\frac{f(y)-f(t)}{y-t}.$$

Then

$$\begin{aligned} \lim_{t\to x+}\frac{f(t)-f(x)}{t-x} &\leq \frac{f(y)-f(x)}{y-x} \\ \frac{f(y)-f(x)}{y-x} &\leq \lim_{t\to y-}\frac{f(y)-f(t)}{y-t}, \end{aligned}$$

and since $f'(x), f'(y)$ both exist, $f'(x)\leq f'(y)$. Conversely, suppose that f' is monotonically increasing. Let $x,y\in(a,b)$ with $x<y$ and let $\lambda\in(0,1)$. Let $t=(1-\lambda)x+\lambda y$.

By the mean value theorem,

$$\frac{f(t)-f(x)}{t-x} = f'(t_1)$$
$$\frac{f(y)-f(t)}{y-t} = f'(t_2)$$

for some $t_1 \in (x,t)$ and $t_2 \in (t,y)$. Since $t_1 < t_2$,

$$\begin{aligned}
\frac{f(t)-f(x)}{t-x} &\le \frac{f(y)-f(t)}{y-t} \\
(1-\lambda)(y-x)(f(t)-f(x)) &\le \lambda(y-x)(f(y)-f(t)) \\
f((1-\lambda)x+\lambda y) &\le (1-\lambda)f(x)+\lambda f(y),
\end{aligned}$$

which shows that f is convex. If f'' is defined on (a,b), then f' is monotonically increasing if and only if $f''(x) \ge 0$ for all $x \in (a,b)$. □

Theorem 72. *[Exercise 5.15] Let $a \in \mathbb{R}$ and suppose that $f : (a,\infty) \to \mathbb{R}$ is twice-differentiable. Suppose that M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ respectively on (a,∞). Then $M_1^2 \le 4M_0M_2$.*

Proof. Let $x \in (a,\infty)$. For any $h > 0$, by Theorem 5.15, there exists a point $\xi \in (x, x+2h)$ such that

$$\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(\xi) \\
f'(x) &= \frac{1}{2h}\left[f(x+2h)-f(x)\right] - hf''(\xi).
\end{aligned}$$

Then

$$\begin{aligned}
|f'(x)| &\le \left|\frac{1}{2h}\left[f(x+2h)-f(x)\right] - hf''(\xi)\right| \\
&\le \frac{|f(x+2h)|+|f(x)|}{2h} + h\,|f''(\xi)| \\
&\le hM_2 + \frac{M_0}{h}
\end{aligned}$$

so that $M_1 \le hM_2 + M_0/h$ since M_1 is the least upper bound of $|f'(x)|$. Setting $h = M_1/(2M_2)$ gives $M_1^2 \le 4M_0M_2$. □

Theorem 73. *[Exercise 5.16] Suppose that $f : (0,\infty) \to \mathbb{R}$ is twice-differentiable, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Then $f'(x) \to 0$ as $x \to \infty$.*

Proof. Choose M such that $|f''(x)| \le M$ for all $x \in (0,\infty)$. Let $\varepsilon > 0$ be given. There exists a A such that $|f(x)| < \varepsilon^2/(16M)$ for all $x \in (A,\infty)$, and by Theorem 72 we have $|f'(x)| \le \varepsilon/2 < \varepsilon$ for all $x \in (A,\infty)$. This shows that $f'(x) \to 0$ as $x \to \infty$. □

Theorem 74. *[Exercise 5.17] Suppose that $f : [-1,1] \to \mathbb{R}$ is a three times differentiable function such that*

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Then $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

Proof. By Theorem 5.15, there exist points $s \in (0,1)$ and $t \in (-1,0)$ such that

$$\begin{aligned} f(1) &= f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6} \\ (*) \qquad 1 &= \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}, \\ f(-1) &= f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6} \\ (**) \qquad 0 &= \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6}. \end{aligned}$$

Subtracting (**) from (*) gives $f^{(3)}(s) + f^{(3)}(t) = 6$. If $f^{(3)}(s) \geq 3$ then we are done; otherwise, $f^{(3)}(s) = 6 - f^{(3)}(t) < 3$, so $f^{(3)}(t) > 3$. □

Theorem 75. *[Exercise 5.18] Let n be a positive integer. Suppose that for $f : [a,b] \to \mathbb{R}$, the value $f^{(n-1)}(t)$ exists for every $t \in [a,b]$. Let α, β, and P be as in Theorem 5.15. Define $Q(t) = (f(t) - f(\beta)) \,/\, (t - \beta)$ for all $t \in [a,b]$ and $t \neq \beta$. Then*

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof. We want to prove that

$$\frac{Q^{(n-1)}(t)}{(n-1)!}(\beta - t)^n = f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(\beta - t)^k$$

for all $n \geq 1$. The case $n = 1$ is equivalent to the definition of Q. Assuming the statement for n and differentiating the above expression, we have

$$\begin{aligned} \frac{Q^{(n)}(t)}{(n-1)!}(\beta - t)^n - \frac{Q^{(n-1)}(t)}{(n-1)!}n(\beta - t)^{n-1} &= -\frac{f^{(n)}(t)}{(n-1)!}(\beta - t)^{n-1} \\ \frac{Q^{(n)}(t)}{(n-1)!}(\beta - t)^{n+1} &= \frac{Q^{(n-1)}(t)}{(n-1)!}n(\beta - t)^n - \frac{f^{(n)}(t)}{(n-1)!}(\beta - t)^n, \end{aligned}$$

where in the first line, most of the terms on the right vanish. Applying the induction hypothesis gives

$$\frac{Q^{(n)}(t)}{(n-1)!}(\beta-t)^{n+1} = nf(\beta) - n\sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(\beta-t)^k - \frac{f^{(n)}(t)}{(n-1)!}(\beta-t)^n$$

$$\frac{Q^{(n)}(t)}{n!}(\beta-t)^{n+1} = f(\beta) - \sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}(\beta-t)^k,$$

which proves the statement for all n. Setting $t=\alpha$ produces the desired result. □

Theorem 76. *[Exercise 5.22(a)] Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $f'(t)\neq 1$ for all $t\in\mathbb{R}$. Then f has at most one fixed point.*

Proof. Suppose that f has two fixed points, $x=f(x)$ and $y=f(y)$, with $x\neq y$. By the mean value theorem, there exists a $c\in(x,y)$ such that

$$\frac{f(y)-f(x)}{y-x} = 1 = f'(c),$$

which is a contradiction. □

Theorem 77. *[Exercise 5.22(b)] Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(t)=t+(1+e^t)^{-1}$. Then f has no fixed point, but $f'(t)\in(0,1)$ for all $t\in\mathbb{R}$.*

Proof. To show that f has no fixed point, note that $(1+e^t)^{-1}\neq 0$ for all $t\in\mathbb{R}$, so that $f(t)=t+(1+e^t)^{-1}\neq t$ for all $t\in\mathbb{R}$. Also,

$$f'(t) = 1-\frac{e^t}{(1+e^t)^2}$$

$$= 1-\frac{1}{1+e^t}+\frac{1}{(1+e^t)^2}.$$

From the first line, $f'(t)<1$ for all $t\in\mathbb{R}$, and from the second line, $f'(t)>0$ for all $t\in\mathbb{R}$. □

Theorem 78. *[Exercise 5.22(c)] Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function. If there exists a constant $A<1$ such that $|f'(t)|\leq A$ for all $t\in\mathbb{R}$, then f has a fixed point $x=\lim_{n\to\infty}x_n$ where $x_0\in\mathbb{R}$ is arbitrary and $x_{n+1}=f(x_n)$ for $n\geq 0$.*

Proof. The case $A=0$ is trivial, so we may assume that $A>0$. By the mean value theorem, $|f(x)-f(y)|\leq A|x-y|$ for all $x,y\in\mathbb{R}$. In particular, $|x_{i+1}-x_{j+1}|\leq A|x_i-x_j|$ for all $i,j\geq 0$, and $|x_m-x_{m-1}|\leq A^{m-1}|x_1-x_0|$ for all $m\geq 1$. We now prove that for all $n\geq 1$,

$$|x_{m+n}-x_m| \leq \frac{A(1-A^n)}{1-A}|x_m-x_{m-1}|.$$

The case $n = 1$ is clear. Assuming the statement for $n - 1$, we have

$$\begin{aligned} |x_{m+n} - x_m| &\leq |x_{m+n-1} - x_m| + |x_{m+n} - x_{m+n-1}| \\ &\leq \frac{A(1 - A^{n-1})}{1 - A} |x_m - x_{m-1}| + A^n |x_m - x_{m-1}| \\ &= \frac{A(1 - A^n)}{1 - A} |x_m - x_{m-1}|, \end{aligned}$$

which proves the statement for all $n \geq 1$. Furthermore,

$$|x_{m+n} - x_m| < \frac{A}{1 - A} |x_m - x_{m-1}|$$

for all $n \geq 1$. Let $\varepsilon > 0$ be given. Recall that $|x_m - x_{m-1}| \leq A^{m-1} |x_1 - x_0|$ for all $m \geq 1$ and that $A < 1$; there exists a N such that $|x_k - x_{k-1}| \leq \varepsilon(1 - A)/A$ for all $k \geq N$. Let $m, n \geq N$ and assume without loss of generality that $m < n$. Then

$$\begin{aligned} |x_n - x_m| &= \left|x_{m+(n-m)} - x_m\right| \\ &< \frac{A}{1 - A} |x_m - x_{m-1}| \\ &< \varepsilon, \end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence. By Theorem 3.11, $\{x_n\}$ converges to some value x; we want to show that x is indeed a fixed point of f. Fix $\varepsilon > 0$. We know that $x_n \to x$, $\{x_n\}$ is a Cauchy sequence, and $f(x_n) \to f(x)$ because f is continuous. Then there exists some integer n such that

$$\begin{aligned} |x - f(x)| &\leq |x - x_n| + |x_n - f(x_n)| + |f(x_n) - f(x)| \\ &= |x - x_n| + |x_n - x_{n+1}| + |f(x_n) - f(x)| \\ &< 3\varepsilon. \end{aligned}$$

Since ε was arbitrary, $x = f(x)$. □

Theorem 79. *[Exercise 5.23] The function* $f(x) = (x^3 + 1)/3$ *has three fixed points* α, β, γ, *where* $-2 < \alpha < -1$, $0 < \beta < 1$, *and* $1 < \gamma < 2$. *For an arbitrarily chosen* x_1, *define* $\{x_n\}$ *by setting* $x_{n+1} = f(x_n)$.

(1) *If* $x_1 < \alpha$, *then* $x_n \to -\infty$ *as* $n \to \infty$.
(2) *If* $\alpha < x_1 < \gamma$, *then* $x_n \to \beta$ *as* $n \to \infty$.
(3) *If* $\gamma < x_1$, *then* $x_n \to +\infty$ *as* $n \to \infty$.

Proof. Let $g(x) = x^3 - 3x + 1$; since α, β, γ are fixed points of f, they are roots of g. Suppose that $x_1 < \alpha$. For any $c > 0$, we can compute

$$\begin{aligned} g(\alpha - c) &= (\alpha^3 - 3\alpha + 1) - 3\alpha^2 c + 3\alpha c^2 - c^3 + 3c \\ &= c(3(1-\alpha^2) + 3\alpha c - c^2) \\ &< 3\alpha c^2 - c^3 \\ &< -c^3 \end{aligned}$$

$$(*) \qquad f(\alpha - c) < (\alpha - c) - \frac{c^3}{3}.$$

Let $d = \alpha - x_1 > 0$; (*) shows that $x_{n+1} < x_n - d/3$ for every $n \geq 1$, and clearly $x_n \to -\infty$ as $n \to \infty$. Now suppose that $\alpha < x_1 < \gamma$. A simple induction argument shows that $\alpha < x_n < \gamma$ for all $n \geq 1$, and by a variation on Theorem 78, $x_n \to \beta$ since $f'(x) = x^2 \in [0, \max(\alpha, \gamma)]$ for all $x \in [\alpha, \gamma]$. Finally, the case for $\gamma < x_1$ is similar to the case $x_1 < \alpha$. □

Proposition 80. *[Exercise 5.25] Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function with $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 < f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. [Note: the inequality $0 \leq f''(x)$ has been changed to $0 < f''(x)$.]*

Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We now prove by induction that $x_{n+1} \in (\xi, x_n)$ for all n. For all n, applying the mean value theorem gives a value $c \in (\xi, x_n)$ such that

$$\frac{f(x_n) - f(\xi)}{x_n - \xi} = f'(c) < f'(x_n),$$

since $c < x_n$ and f' is strictly increasing. Therefore

$$\frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1} < x_n - \xi$$

and $\xi < x_{n+1}$. Also, $f(x_n) > 0$ for otherwise $f(y) = 0$ for some $y \in [x_n, b)$ by the intermediate value theorem. Therefore $f(x_n)/f'(x_n) > 0$ and $x_{n+1} < x_n$.

Applying Taylor's theorem with $\alpha = x_n$, $\beta = \xi$ gives a point $t_n \in (\xi, x_n)$ such that

$$\begin{aligned} f(\xi) &= f(x_n) + f'(x_n)(\xi - x_n) + \frac{1}{2} f''(t_n)(\xi - x_n)^2 \\ x_n - \frac{f(x_n)}{f'(x_n)} &= \xi + \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2. \end{aligned}$$

Consider the statement

$$x_{n+1} - \xi \leq \frac{1}{A} \left[A(x_1 - \xi)\right]^{2^n}.$$

If $n = 1$, then

$$\begin{aligned} x_2 - \xi &= \frac{f''(t_n)}{2f'(x_1)}(x_1 - \xi)^2 \\ &\leq \frac{M}{2f'(x_1)}(x_1 - \xi)^2 \\ &< A(x_1 - \xi)^2. \end{aligned}$$

Otherwise, assuming the statement for $n - 1$, we have

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &< A(x_n - \xi)^2 \\ &< \frac{1}{A} \left[A(x_1 - \xi)\right]^{2^{n+1}}, \end{aligned}$$

which proves the statement for all n. Since $0 < x_{n+1} - \xi$ for all n, this shows that $x_n \to \xi$ as $n \to \infty$. Let $g(x) = x - f(x)/f'(x)$. Since ξ is a root of f, $g(\xi) = \xi$, and $x_n \to \xi$, the process amounts to finding a fixed point of g. For x near ξ,

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)^2 - f''(x)f(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\approx 0. \end{aligned}$$

Theorem 81. *[Exercise 5.26] Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function with $f(a) = 0$. Let A be a real number such that $|f'(x)| \leq A|f(x)|$ for all $x \in [a, b]$. Then $f = 0$.*

Proof. Let $x_0 \in [a,b]$, $M_0 = \sup_{a\le x\le x_0} |f(x)|$, and $M_1 = \sup_{a\le x\le x_0} |f'(x)|$. By the mean value theorem, there exists a point $c \in (a, x_0)$ such that

$$\begin{aligned}\frac{f(x_0)}{x_0 - a} &= f'(c)\\ |f(x_0)| &\le M_1(x_0 - a) \le A(x_0 - a)M_0.\end{aligned}$$

Suppose that $x_0 > a$ and let $x \in (a, x_0)$. By the mean value theorem, there exists a point $c \in (a, x)$ such that

$$\begin{aligned}\frac{f(x)}{x - a} &= f'(c)\\ |f(x)| &\le M_1(x - a)\\ &\le M_1(x_0 - a) \le A(x_0 - a)M_0.\end{aligned}$$

Since $f(a) = 0$, we have $|f(x)| \le M_1(x_0 - a) \le A(x_0 - a)M_0$ for all $x \in [a,b]$. Suppose that $A(x_0 - a) < 1$; then $M_0 = 0$ for otherwise $A(x_0 - a)M_0 < M_0$ is a lower bound of $|f(x)|$ in $[a, x_0]$, which contradicts the definition of M_0. Therefore, if $x_0 > a$ is small enough, then $f(x) = 0$ for all $x \in [a, x_0]$. Now divide the interval $[a,b]$ into n closed intervals $[a, p_1], [p_1, p_2], \dots, [p_n, b]$ where n is the smallest integer with $n(x_0 - a) \ge b - a$, and $p_k = a + k(x_0 - a)$. We have shown that f is zero on $[a, x_0] = [a, p_1]$; since $f(p_1) = 0$, applying the argument on $[p_1, p_2]$ shows that f is zero on $[p_1, p_2]$, and so on. □

Theorem 82. *[Exercise 5.27] Let R be a rectangle in the plane given by $a \le x \le b$ and $\alpha \le y \le \beta$ for $(x,y) \in R$. Let $\phi : R \to \mathbb{R}$ be a function defined on the rectangle. A* ***solution*** *of the initial-value problem*

$$y' = \phi(x,y), \quad y(a) = c \quad \text{where } \alpha \le c \le \beta$$

is by definition a differentiable function $f : [a,b] \to [\alpha, \beta]$ such that $f(a) = c$ and $f'(x) = \phi(x, f(x))$ for all $x \in [a,b]$. Suppose that there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \le A\,|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$. Then the problem has at most one solution.

Proof. Let f, g be two solutions of the initial-value problem, and let $h : [a,b] \to \mathbb{R}$ be given by $h(x) = f(x) - g(x)$. Then

$$\begin{aligned}|h'(x)| &= |f'(x) - g'(x)|\\ &= |\phi(x, f(x)) - \phi(x, g(x))|\\ &\le A\,|f(x) - g(x)|\\ &= A\,|h(x)|\end{aligned}$$

for all $x \in [a,b]$. Since $h(a) = 0$, by Theorem 81, $h = 0$ and $f = g$. □

Chapter 6. The Riemann-Stieltjes Integral

Theorem 83. *[Exercise 6.1] Suppose* $\alpha : [a, b] \to \mathbb{R}$ *is increasing,* $a \leq x_0 \leq b$, α *is continuous at* x_0, $f(x_0) = 1$, *ad* $f(x) = 0$ *if* $x \neq x_0$. *Then* $f \in \mathcal{R}(\alpha)$ *and* $\int_a^b f\, d\alpha = 0$.

Proof. By Theorem 6.10, $f \in \mathcal{R}(\alpha)$ since f has only one point of discontinuity. Also, since $L(P, f, \alpha) = 0$ for all partitions P, $\int_a^b f\, d\alpha = 0$. □

Theorem 84. *[Exercise 6.2] Suppose* $f : [a, b] \to \mathbb{R}$ *is a continuous function,* $f \geq 0$, *and* $\int_a^b f(x)\, dx = 0$. *Then* $f = 0$.

Proof. Suppose that $f \neq 0$; we can choose $x_0 \in (a, b)$ such that $f(x_0) > 0$, for f cannot be nonzero only at its endpoints due to continuity. Then there exists a $\delta > 0$ such that $|f(x_0) - f(x)| < f(x_0)/2$ whenever $|x_0 - x| < \delta$. In particular, $f(x) > f(x_0)/2$ for all $x \in [x_0 - \gamma, x_0 + \gamma]$, where $\gamma = \min\{\delta/2, x_0 - a, b - x_0\}$. By Theorem 6.12,

$$\begin{aligned}
\int_a^b f(x)\, dx &= \int_a^{x_0-\gamma} f(x)\, dx + \int_{x_0-\gamma}^{x_0+\gamma} f(x)\, dx + \int_{x_0+\gamma}^b f(x)\, dx \\
&\geq \int_{x_0-\gamma}^{x_0+\gamma} f(x)\, dx \\
&\geq \int_{x_0-\gamma}^{x_0+\gamma} f(x_0)/2\, dx \\
&> 0,
\end{aligned}$$

which is a contradiction. Therefore $f = 0$. □

Theorem 85. *[Exercise 6.3] Define three functions* β_1, β_2, β_3 *as follows:* $\beta_j(x) = 0$ *if* $x < 0$, $\beta_j(x) = 1$ *if* $x > 0$ *for* $j = 1, 2, 3$*; and* $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_2(0) = \frac{1}{2}$. *Let* f *be a bounded function on* $[-1, 1]$.

(1) $f \in \mathcal{R}(\beta_1)$ *if and only if* $f(0+) = f(0)$, *and then* $\int_{-1}^1 f(x)\, d\beta_1 = f(0)$.
(2) $f \in \mathcal{R}(\beta_2)$ *if and only if* $f(0-) = f(0)$, *and then* $\int_{-1}^1 f(x)\, d\beta_2 = f(0)$.
(3) $f \in \mathcal{R}(\beta_3)$ *if and only if* f *is continuous at 0.*
(4) *If* f *is continuous at 0 then*

$$\int_{-1}^1 f(x)\, d\beta_1 = \int_{-1}^1 f(x)\, d\beta_2 = \int_{-1}^1 f(x)\, d\beta_3 = f(0).$$

Proof. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon/2$ whenever $0 < x < \delta$. Let $\gamma = \min(1, \delta)/2$ and let $P = \{-1, 0, \gamma, 1\}$ be a partition of $[-1, 1]$. Then

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= \sup_{x \in [0,\gamma]} f(x) - \inf_{x \in [0,\gamma]} f(x) \\ &< \varepsilon, \end{aligned}$$

so $f \in \mathcal{R}(\beta_1)$. Furthermore,

$$\begin{aligned} U(P, f, \beta_1) &= \sup_{x \in [0,\gamma]} f(x) \\ &\leq f(0) + \frac{\varepsilon}{2}, \end{aligned}$$

which shows that $\int_{-1}^{1} f(x)\, d\beta_1 = f(0)$ since ε was arbitrary. Conversely, suppose that $f \in \mathcal{R}(\beta_1)$. Let $\varepsilon > 0$ be given. There exists a partition P of $[-1, 1]$ such that

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_i - m_i \\ &< \varepsilon \end{aligned}$$

for some i with $x_{i-1} \leq 0 < x_i$, where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. Then whenever $0 < t < x_i$ we have $0 \leq f(t) - m_i < \varepsilon$ and $-\varepsilon < m_i - f(0) \leq 0$ so that $|f(t) - f(0)| < \varepsilon$. This shows that $f(0+) = f(0)$. The proof is similar for (2) and (3). □

Theorem 86. *[Exercise 6.4] If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, then $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.*

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. For all $x < y$ there exist both rational and irrational numbers in (x, y), so $M_i = 1$ and $m_i = 0$ for every i. Therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} \triangle x_i \\ &= b - a, \end{aligned}$$

and $f \notin \mathcal{R}$ on $[a, b]$. □

Remark 87. [Exercise 6.5] Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Assume that $a < b$ and let $f(x) = 1$ if $x \in \mathbb{Q}$, $f(x) = -1$ if $x \notin \mathbb{Q}$. Then $f^2 \in \mathcal{R}$ with $\int_a^b f(x)^2\, dx = b - a$, but $f \notin \mathcal{R}$. This disproves the first part of the statement. However, the second statement is true by Theorem 6.11, since $x \mapsto x^{1/3}$ is continuous on any interval in $\mathbb{R}$.

Theorem 88. *[Exercise 6.7] Let $f : (0,1] \to \mathbb{R}$ and suppose that $f \in \mathcal{R}$ on $[c,1]$ for every $c > 0$. Define*

$$\int_0^1 f(x)\,dx = \lim_{c\to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(1) *If $f \in \mathcal{R}$ on $[0,1]$, then this definition of the integral agrees with the old one.*
(2) *There exists a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.*

Proof. If $f \in \mathcal{R}$ on $[0,1]$, then by Theorem 6.20, $F(c) = \int_c^1 f(x)\,dx$ is continuous on $[0,1]$. Therefore $\lim_{c\to 0} F(c) = \int_0^1 f(x)\,dx$. $\square$

Theorem 89. *[Exercise 6.8] Suppose that $f \in \mathcal{R}$ on $[a,b]$ for every $b > a$ where a is fixed. Define*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1,\infty)$. Then $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

Proof. Suppose that $\int_1^\infty f(x)\,dx$ converges to L. For every $\varepsilon > 0$, there exists a $M \geq 1$ such that $\left|\int_1^b f(x)\,dx - L\right| < \varepsilon/2$ whenever $b \geq M$. Then for all $n \geq m \geq M+1$, we have

$$\begin{aligned}\int_{m-1}^n f(x)\,dx &\leq \int_1^n f(x)\,dx - L + L - \int_1^{m-1} f(x)\,dx \\ &< \varepsilon.\end{aligned}$$

But since f decreases monotonically on $[1,\infty)$,

$$\begin{aligned}0 \leq \sum_{k=m}^n f(k) &\leq \sum_{k=m}^n \int_{k-1}^k f(x)\,dx \\ &= \int_{m-1}^n f(x)\,dx \\ &< \varepsilon,\end{aligned}$$

which shows that $\sum_{n=1}^\infty f(n)$ converges. Conversely, suppose that $\sum_{n=1}^\infty f(n)$ converges; we first show that the sequence $\left\{\int_1^i f(x)\,dx\right\}$ converges. Let $\varepsilon > 0$. There exists a

$M \geq 1$ such that for all $n \geq m \geq M$ we have $0 \leq \sum_{k=m}^{n} f(k) < \varepsilon$. Then for all $m, n \geq M$, assume $m \leq n$ so that

$$\begin{aligned} 0 \leq \int_1^n f(x)\,dx - \int_1^m f(x)\,dx &= \int_m^n f(x)\,dx \\ &= \sum_{k=m}^{n-1} \int_k^{k+1} f(x)\,dx \\ &\leq \sum_{k=m}^{n-1} f(k) \\ &< \varepsilon. \end{aligned}$$

This shows that $\int_1^i f(x)\,dx \to L$ for some $L \geq 0$, and furthermore, $\int_1^i f(x)\,dx \leq L$ for all $i \geq 1$ since the sequence is monotonically increasing. Let $\varepsilon > 0$ be given; there exists a $N \geq 1$ such that $0 \leq L - \int_1^i f(x)\,dx < \varepsilon$ whenever $i \geq N$. Now for all real $b \geq N+1$,

$$\begin{aligned} \int_1^{\lfloor b \rfloor} f(x)\,dx &\leq \int_1^b f(x)\,dx \\ 0 \leq L - \int_1^b f(x)\,dx &\leq L - \int_1^{\lfloor b \rfloor} f(x)\,dx \\ &< \varepsilon. \end{aligned}$$

This proves that $\int_1^\infty f(x)\,dx$ converges to L. □

Theorem 90. *[Exercise 6.9] Suppose that F and G are differentiable on $[a, b]$ for every $b > a$, $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$. If*

$$\lim_{b \to \infty} F(b)G(b)$$

exists (with a finite value) and

$$\int_a^\infty f(x)G(x)\,dx$$

converges, then

$$\int_a^\infty F(x)g(x)\,dx = \lim_{b \to \infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)\,dx.$$

Proof. For every $b > a$,

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

The result follows from Theorem 4.4. □

Theorem 91. *[Exercise 6.10] Let p and q be positive real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(1) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q},$$

with equality if $u^p = v^q$.

(2) *If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and*

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

then

$$\int_a^b fg \, d\alpha \leq 1.$$

(3) *If f and g are complex functions in $\mathcal{R}(\alpha)$, then*

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \ d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \ d\alpha \right\}^{1/q}.$$

Proof. We have

$$\begin{aligned} uv &= (u^p)^{1/p}(v^q)^{1/q} \\ &= \exp\left(\frac{1}{p}\log u^p\right)\exp\left(\frac{1}{q}\log v^q\right) \\ &= \exp\left(\frac{1}{p}\log u^p + \frac{1}{q}\log v^q\right) \\ &\leq \frac{1}{p}\exp\left(\log u^p\right) + \frac{1}{q}\exp\left(\log v^q\right) \\ &= \frac{u^p}{p} + \frac{v^q}{q} \end{aligned}$$

since $1/q = 1 - 1/p$ and exp is convex. If $u^p = v^q$, then

$$\begin{aligned} uv &= (u^p)^{1/p}(v^q)^{1/q} \\ &= (u^p)^{1/p+1/q} \\ &= u^p \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

This proves (1). Let $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$ with $f \geq 0$, $g \geq 0$ and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha.$$

Then (on $[a, b]$)

$$\begin{aligned} fg &\leq \frac{f^p}{p} + \frac{g^q}{q} \\ \int_a^b fg \, d\alpha &\leq \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha \\ &= 1, \end{aligned}$$

which proves (2). Now suppose that f and g are functions in $\mathcal{R}(\alpha)$. Let

$$A = \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p},$$

$$B = \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}$$

so that

$$\int_a^b \left(\frac{|f|}{A} \right)^p d\alpha = 1 = \int_a^b \left(\frac{|g|}{B} \right)^q d\alpha$$

assuming that $A, B > 0$. Applying (2) gives

$$\int_a^b \frac{|f| \, |g|}{AB} \, d\alpha \leq 1,$$

and then

$$\begin{aligned} \left| \int_a^b fg \, d\alpha \right| &\leq \int_a^b |f| \, |g| \, d\alpha \\ &\leq AB \\ &= \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}, \end{aligned}$$

which proves (3). □

Theorem 92. *[Exercise 6.11] Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define*

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$. Then

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Proof. On $[a,b]$ we have

$$\begin{aligned}(f-h)^2 &= (f-g+g-h)^2\\ &= (f-g)^2+2(f-g)(g-h)+(g-h)^2\\ \int_a^b |f-h|^2\ d\alpha &= \int_a^b |f-g|^2\ d\alpha + 2\int_a^b (f-g)(g-h)\,d\alpha + \int_a^b |g-h|^2\ d\alpha\\ &\le \int_a^b |f-g|^2\ d\alpha + 2\left|\int_a^b (f-g)(g-h)\,d\alpha\right| + \int_a^b |g-h|^2\ d\alpha.\end{aligned}$$

Applying Theorem 91 gives

$$\begin{aligned}\|f-h\|_2^2 &\le \|f-g\|_2^2 + 2\left\{\int_a^b |f-g|^2\ d\alpha\right\}^{1/2}\left\{\int_a^b |g-h|^2\ d\alpha\right\}^{1/2} + \|g-h\|_2^2\\ &= \|f-g\|_2^2 + 2\,\|f-g\|_2\,\|g-h\|_2 + \|g-h\|_2^2\\ &= \left(\|f-g\|_2 + \|g-h\|_2\right)^2,\end{aligned}$$

which completes the proof. □

Theorem 93. *[Exercise 6.12] Suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Then there exists a continuous function g on $[a,b]$ such that $\|f-g\|_2 < \varepsilon$.*

Proof. Let $M = \sup f(x)$ and $m = \inf f(x)$ over $x \in [a,b]$, and assume that $M \neq m$ for otherwise f is constant and the result follows by setting $g = f$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon^2/(M-m)$. Define

$$g(t) = \frac{x_i - t}{\triangle x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\triangle x_i} f(x_i)$$

for $x_{i-1} \le t \le x_i$; g is continuous at each x_i. For each i, let $M_i = \sup f(x)$ and $m_i = \inf f(x)$, over $x \in [x_{i-1}, x_i]$. We can rewrite g as

$$g(t) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{\triangle x_i}(t - x_{i-1}),$$

which shows that

$$m \le m_i \le g(x) \le M_i \le M$$

for all $x \in [a,b]$. Then

$$\begin{aligned}
\|f-g\|_2^2 &= \int_a^b |f(x)-g(x)|^2 \; d\alpha \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x)-g(x)|^2 \; d\alpha \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |M_i - m_i|^2 \; d\alpha \\
&= \sum_{i=1}^n |M_i - m_i|^2 \triangle\alpha_i \\
&\le (M-m)\sum_{i=1}^n (M_i - m_i)\triangle\alpha_i \\
&= (M-m)(U(P,f,\alpha) - L(P,f,\alpha)) \\
&< \varepsilon^2,
\end{aligned}$$

which completes the proof. □

Theorem 94. *[Exercise 6.13] Define*

$$f(x) = \int_x^{x+1} \sin(t^2)\, dt.$$

(1) $|f(x)| < 1/x$ *if* $x > 0$.
(2) $2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$ *where* $|r(x)| < c/x$ *and* c *is a constant.*
(3) $\limsup_{x\to\infty} xf(x) = 1$ *and* $\liminf_{x\to\infty} xf(x) = -1$.
(4) $\int_0^\infty \sin(t^2)\, dt$ *converges.*

Proof. Let $x > 0$. By Theorem 6.8,

$$u \mapsto \frac{\sin(u)}{2\sqrt{u}}$$

is Riemann-integrable on $[x^2, (x+1)^2]$. Let $\varphi : [x, x+1] \to [x^2, (x+1)^2]$ be given by $t \mapsto t^2$. Since φ strictly increasing and onto, applying Theorem 6.19 gives

$$\int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}}\, du = \int_x^{x+1} \sin(t^2)\, dt = f(x).$$

Let $F(u) = -\cos u$ and $G(u) = 1/(2\sqrt{u})$ so that $F'(u) = \sin u$ and $G'(u) = -1/(4u^{3/2})$. By Theorem 6.22,

$$\begin{aligned}
f(x) &= \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}\,du \\
&\le \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}}\,du \\
&= \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\
&= \frac{\cos(x^2)+1}{2x} - \frac{\cos[(x+1)^2]+1}{2(x+1)} \\
&\le \frac{1}{x} - \frac{1}{x+1} \\
&< \frac{1}{x},
\end{aligned}$$

and similarly replacing $\cos u$ with 1 gives $-1/x < f(x)$. This proves (1). For (2),

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x) \tag{*}$$

where

$$r(x) = \frac{1}{x+1}\cos[(x+1)^2] - x\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\,du.$$

Furthermore,

$$\begin{aligned}
|r(x)| &\le \frac{1}{x+1} + x\int_{x^2}^{(x+1)^2} \frac{1}{2u^{3/2}}\,du \\
&= \frac{1}{x+1} + x\left(\frac{1}{x} - \frac{1}{x+1}\right) \\
&= \frac{2}{1+x} \\
&< \frac{2}{x}
\end{aligned}$$

since $2x < 2 + 2x$. Then equation (*) shows (3). The integral $\int_0^\infty \sin(t^2)\,dt$ converges if $\int_1^\infty \sin(t^2)\,dt$ converges. As in (1) we have for all $b > 1$,

$$\int_1^b \sin(t^2)\,dt = \int_1^{b^2} \frac{\sin(u)}{2\sqrt{u}}\,du$$

and

$$\int_1^{b^2} \frac{\sin(u)}{2\sqrt{u}}\,du = -\frac{\cos(b^2)}{2b} + \frac{\cos 1}{2} - \int_1^{b^2} \frac{\cos u}{4u^{3/2}}\,du.$$

Since $\int_1^\infty 1/(4u^{3/2})\, du$ converges, applying Theorem 90 shows that $\int_0^\infty \sin(t^2)\, dt$ converges. □

Theorem 95. *[Exercise 6.15] Suppose that $f : [a,b] \to \mathbb{R}$ is a continuously differentiable function with $f(a) = f(b) = 0$, and*

$$\int_a^b f(x)^2\, dx = 1.$$

Then

$$\int_a^b x f(x) f'(x)\, dx = -\frac{1}{2}$$

and

$$\left(\int_a^b f'(x)^2\, dx\right)\left(\int_a^b x^2 f(x)^2\, dx\right) > \frac{1}{4}.$$

Proof. Let $F(x) = f(x)$ and $G(x) = xf(x)$ so that $F'(x) = f'(x)$ and $G'(x) = xf'(x) + f(x)$. By Theorem 6.22,

$$\begin{aligned}\int_a^b x f(x) f'(x)\, dx &= -\int_a^b f(x)[xf'(x) + f(x)]\, dx \\ &= -\int_a^b f(x)^2\, dx - \int_a^b x f(x) f'(x)\, dx \\ &= -\frac{1}{2}.\end{aligned}$$

By Theorem 91 we have

$$\frac{1}{4} = \left|\int_a^b [f'(x)][xf(x)]\, dx\right|^2 \leq \left(\int_a^b |f'(x)|^2\ dx\right)\left(\int_a^b |xf(x)|^2\ dx\right).$$

□

Theorem 96. *[Exercise 6.16] For $1 < s < \infty$, define*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

(1) $\zeta(s) = s \displaystyle\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}\, dx.$

(2) $\zeta(s) = \dfrac{s}{s-1} - s \displaystyle\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx.$

Proof. For every positive integer N,

$$\begin{aligned}
s\int_1^N \frac{\lfloor x\rfloor}{x^{s+1}}\,dx &= s\sum_{n=1}^{N-1}\int_n^{n+1}\frac{\lfloor x\rfloor}{x^{s+1}}\,dx \\
&= s\sum_{n=1}^{N-1} n\int_n^{n+1}\frac{1}{x^{s+1}}\,dx \\
&= \sum_{n=1}^{N-1} n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right) \\
&= \sum_{n=1}^{N-1}\left(\frac{1}{n^{s-1}}-\frac{n+1}{(n+1)^s}+\frac{1}{(n+1)^s}\right) \\
&= \sum_{n=1}^{N-1}\left(\frac{1}{n^{s-1}}-\frac{1}{(n+1)^{s-1}}\right)+\sum_{n=2}^{N}\frac{1}{n^s} \\
&= 1-\frac{1}{N^{s-1}}+\sum_{n=2}^{N}\frac{1}{n^s} \\
&= \sum_{n=1}^{N}\frac{1}{n^s}-\frac{1}{N^{s-1}}
\end{aligned}$$

so that

$$s\int_1^\infty \frac{\lfloor x\rfloor}{x^{s+1}}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

since $s-1>0$. For (2), we have

$$\begin{aligned}
\frac{s}{s-1}-s\int_1^N\frac{x-\lfloor x\rfloor}{x^{s+1}}\,dx &= \frac{s}{s-1}-s\int_1^N\frac{1}{x^s}\,dx+s\int_1^N\frac{\lfloor x\rfloor}{x^{s+1}}\,dx \\
&= \sum_{n=1}^{N}\frac{1}{n^s}+\left(\frac{s}{s-1}\right)\frac{1}{N^{s-1}}-\frac{1}{N^{s-1}}
\end{aligned}$$

and again,

$$\frac{s}{s-1}-s\int_1^\infty\frac{x-\lfloor x\rfloor}{x^{s+1}}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

if $s > 1$. In fact, the integral in (2) converges for all $s > 0$ since

$$\begin{aligned}\int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx &\leq \int_1^N \frac{1}{x^{s+1}}\,dx \\ &= \frac{1}{s}\left(1 - \frac{1}{N^s}\right).\end{aligned}$$

□

Lemma 97. *Suppose that $f \in \mathcal{R}$ on $[a,b]$ and let P be a partition of $[a,b]$. Let c be a real number. If $U(P^*, f, \alpha) \geq c$ for every refinement P^* of P, then $\int_a^b f\,d\alpha \geq c$. If $L(P^*, f, \alpha) \leq c$ for every refinement P^* of P, then $\int_a^b f\,d\alpha \leq c$.*

Proof. Let $\varepsilon > 0$. There exists a partition P' of $[a,b]$ such that

$$U(P', f, \alpha) < \int_a^b f\,d\alpha + \varepsilon.$$

Let $P^* = P \cup P'$; since P^* is a refinement of P, we have

$$2 \leq U(P^*, f, \alpha) \leq U(P', f, \alpha) < \int_a^b f\,d\alpha + \varepsilon,$$

which completes the proof since $\varepsilon > 0$ was arbitrary. The case for the lower sums is analogous. □

Theorem 98. *[Exercise 6.17] Suppose α increases monotonically on $[a,b]$, g is continuous, and $g(x) = G'(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)\,d\alpha.$$

Proof. Let $\varepsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $U(P,g) - L(P,g) < \varepsilon$. Applying the mean value theorem gives points $t_i \in (x_{i-1}, x_i)$ such that $g(t_i)\triangle x_i = G(x_i) - G(x_{i-1})$. Then

$$\begin{aligned}\sum_{i=1}^n \alpha(x_i)g(t_i)\triangle x_i &= \sum_{i=1}^n \alpha(x_i)\left[G(x_i) - G(x_{i-1})\right] \\ &= \sum_{i=2}^{n+1} \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1}) \\ &= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\triangle\alpha_i \qquad (*)\end{aligned}$$

and

$$\sum_{i=1}^{n} |g(x_i) - g(t_i)| \triangle x_i < \varepsilon$$

by Theorem 6.7 so that

$$\begin{aligned}\left|\sum_{i=1}^{n} \alpha(x_i) g(x_i) \triangle x_i - \sum_{i=1}^{n} \alpha(x_i) g(t_i) \triangle x_i\right| &= \left|\sum_{i=1}^{n} \alpha(x_i) \left[g(x_i) - g(t_i)\right] \triangle x_i\right| \\ &\leq \sum_{i=1}^{n} \left|\alpha(x_i) \left[g(x_i) - g(t_i)\right]\right| \triangle x_i \\ &\leq M\varepsilon\end{aligned}$$

where $M = \sup \alpha(x)$ over $x \in [a, b]$. From (*) we have

$$\begin{aligned}\sum_{i=1}^{n} \alpha(x_i) g(x_i) \triangle x_i &\leq G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \triangle \alpha_i + M\varepsilon \\ L(P, \alpha g) + L(P, G, \alpha) &\leq G(b)\alpha(b) - G(a)\alpha(a) + M\varepsilon\end{aligned}$$

and similarly

$$G(b)\alpha(b) - G(a)\alpha(a) - M\varepsilon \leq U(P, \alpha g) + U(P, G, \alpha).$$

But these two inequalities are true for any refinement of P, so by Theorem 97,

$$S - M\varepsilon \leq \overline{\int_a^b} \alpha(x) g(x)\, dx = \underline{\int_a^b} \alpha(x) g(x)\, dx \leq S + M\varepsilon$$

where

$$S = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)\, d\alpha.$$

Since ε was arbitrary, the result follows. □

Theorem 99. *[Exercise 6.18] Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by*

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi i t \sin(1/t)}.$$

(1) *γ_1, γ_2 are rectifiable. γ_1 has length 2π and γ_2 has length 4π.*
(2) *γ_3 is not rectifiable.*

Proof. Applying Theorem 6.27 shows that

$$\begin{aligned}\Lambda(\gamma_1) &= \int_0^{2\pi} \left|ie^{it}\right|\, dt \\ &= 2\pi\end{aligned}$$

and

$$\Lambda(\gamma_1) = \int_0^{2\pi} \left|2ie^{2it}\right| \, dt$$
$$= 4\pi.$$

Let $P = \{x_{2n+1}, \ldots, 2/\pi\}$ with

$$x_k = \frac{2}{(2k+1)\pi}$$

so that

$$\begin{aligned}\Lambda(P, \gamma_3) &= \sum_{k=1}^{2n+1} \left|e^{2\pi i x_k \sin(1/x_k)} - e^{2\pi i x_{k-1} \sin(1/x_{k-1})}\right|. \\ &\geq \sum_{k=1}^{n} \left|e^{4i/(4k+1)} - e^{-4i/(4k-1)}\right| \\ &= \sum_{k=1}^{n} \sqrt{2 - 2\cos\left(\frac{4}{4k+1} + \frac{4}{4k-1}\right)} \\ &\to \infty\end{aligned}$$

as $n \to \infty$ since $\sqrt{2 - 2\cos x} = x + O(x^3)$ and

$$\sum_{k=1}^{\infty} \left(\frac{4}{4k+1} + \frac{4}{4k-1}\right)$$

diverges. This shows that $\Lambda(\gamma_3) = +\infty$ and therefore γ_3 is not rectifiable. □

Theorem 100. *[Exercise 6.19] Let $\gamma_1 : [a, b] \to \mathbb{R}^k$ be a curve and let $\phi : [c, d] \to [a, b]$ be a continuous bijection such that $\phi(c) = a$. Define $\gamma_2 = \gamma_1 \circ \phi$. Then:*

(1) *γ_2 is an arc if and only if γ_1 is an arc.*
(2) *γ_2 is a closed curve if and only if γ_1 is a closed curve.*
(3) *γ_2 is rectifiable if and only if γ_1 is rectifiable, and in that case γ_1, γ_2 have the same length.*

Proof. (1) is clear since the composition of injections is also an injection (ϕ, ϕ^{-1} are both injective). (2) is clear since ϕ is monotonically increasing and $\phi(d) = b$. For (3), suppose that γ_1 is rectifiable. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[c, d]$. Define $P' = \{\phi(x_0), \ldots, \phi(x_n)\}$; This is a well-defined partition of $[a, b]$, for ϕ must be monotonically increasing. Then

$$\begin{aligned}\Lambda(P,\gamma_2) &= \sum_{i=1}^{n} |\gamma_1(\phi(x_i)) - \gamma_1(\phi(x_{i-1}))| \\ &= \Lambda(P',\gamma_1) \\ &\le \Lambda(\gamma_1).\end{aligned}$$

Since this holds for all partitions, we have $\Lambda(\gamma_2) \le \Lambda(\gamma_1)$ which shows that γ_2 is rectifiable. Noting that $\gamma_1 = \gamma_2 \circ \phi^{-1}$, the same argument proves that $\Lambda(\gamma_1) \le \Lambda(\gamma_2)$. □

Chapter 7. Sequences and Series of Functions

Theorem 101. *[Exercise 7.1] Every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof. Let $f_n \to f$ uniformly on E, where each f_n is bounded. That is, for each n, $M_n = \sup_{x \in E} |f_n(x)|$ is finite. Choose an integer N such that $|f_n(x) - f(x)| < 1$ for all $n \ge N$ and $x \in E$. In particular,

$$\begin{aligned}|f(x)| &\le |f_N(x) - f(x)| + |f_N(x)| \\ &< M_N + 1\end{aligned}$$

for all $x \in E$, and

$$\begin{aligned}|f_n(x)| &\le |f_n(x) - f(x)| + |f(x)| \\ &< M_N + 2\end{aligned}$$

for all $n \ge N$. Take $M = \max\{M_1, \ldots, M_{N-1}, M_N + 2\}$ so that $|f_n(x)| \le M$ for all $n \ge 1$. This completes the proof. □

Theorem 102. *[Exercise 7.2] If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, then $\{f_n + g_n\}$ converges uniformly on E. Furthermore, if $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then $\{f_n g_n\}$ converges uniformly on E.*

Proof. Let $f_n \to f$ and $g_n \to g$ uniformly on E. For any $\varepsilon > 0$, there exist integers N_1, N_2 such that for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon/2$ whenever $n \ge N_1$ and $|g_n(x) - g(x)| < \varepsilon/2$ whenever $n \ge N_2$. Then $|f_n(x) - f(x) + g_n(x) - g(x)| < \varepsilon$ whenever $x \in E$ and $n \ge \max(N_1, N_2)$, which shows that $f_n + g_n \to f + g$ uniformly on E. Now suppose that $\{f_n\}, \{g_n\}$ are sequences of bounded functions, so that f, g are bounded. Let $\varepsilon > 0$ be given. Choose N_1, N_2 such that for all $x \in E$, $|f_n(x) - f(x)| < \sqrt{\varepsilon}$ whenever $n \ge N_1$ and $|g_n(x) - g(x)| < \sqrt{\varepsilon}$ whenever $n \ge N_2$. Then for all $x \in E$ and $n \ge \max(N_1, N_2)$,

$$|[f_n(x) - f(x)][g_n(x) - g(x)]| < \varepsilon,$$

which shows that $(f_n - f)(g_n - g) \to 0$ uniformly on E. Since f, g are bounded, $f(g_n - g) \to 0$ and $g(f_n - f) \to 0$ uniformly on E, so that

$$\begin{aligned} f_n g_n - fg &= (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f) \\ &\to 0 \end{aligned}$$

uniformly on E. □

Theorem 103. *[Exercise 7.3] Let $f_n(x) = x$ and $g_n(x) = 1/n$; $f_n \to x$ and $g_n \to 0$ uniformly on $\mathbb{R}$, but $\{f_n g_n\}$ does not converge uniformly.*

Proof. Choose $\varepsilon = 1$ and let N be an integer. Then $(f_n g_n)(N) \geq 1 = \varepsilon$ for all $n \geq N$, which shows that $\{f_n g_n\}$ does not converge uniformly. □

Example 104. [Exercise 7.4] Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

- The series does not converge when $x = 0$, and is undefined when $x = -1/n^2$ for any integer $n \geq 1$. However, it converges absolutely for all other x.
- The series converges uniformly on a set E if and only if $0, -1, -1/2^2, -1/3^2, \ldots$ are all interior points of E^c.
- f is continuous and bounded on any set where it converges uniformly.

Example 105. [Exercise 7.5] Let

$$f_n(x) = \begin{cases} 0 & \text{for } x < \frac{1}{n+1}, \\ \sin^2 \frac{\pi}{x} & \text{for } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ 0 & \text{for } \frac{1}{n} < x. \end{cases}$$

For any x, there exists a N such that $1/n < x$ for all $n \geq N$; this shows that $f_n \to 0$. Choose $\varepsilon = 1$; then for all N we have

$$\begin{aligned} f_N\left(\frac{(2N+1)}{2N(N+1)}\right) &= \sin^2(2N(N+1)\pi/(2N+1)) \\ &= 1, \end{aligned}$$

which shows that $\{f_n\}$ does not converge uniformly. Now consider the series $\sum f_n(x)$. For any x there are only finitely many non-zero terms, so that the series converges absolutely for all x. Again, the series fails to converge uniformly.

Theorem 106. *[Exercise 7.6] The series*

$$(*)\qquad \sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2} = \sum_{n=1}^{\infty}(-1)^n\left(\frac{x^2}{n^2}+\frac{1}{n}\right)$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let I be a bounded interval and let $M = \sup_{x\in I}|x|$. By Theorem 3.43, $\sum(-1)^n/n$ converges, and $\sum(-1)^n x^2/n^2$ converges (absolutely) for all x. Therefore (*) converges, and it remains to show that the convergence is uniform. Let $\varepsilon > 0$ be given and choose N_1 such that

$$\left|\sum_{k=m}^{n}(-1)^n\frac{1}{n}\right| < \varepsilon/2$$

whenever $n \geq m \geq N_1$. Also choose N_2 such that

$$\sum_{k=m}^{n}\frac{M^2}{n^2} < \varepsilon/2$$

whenever $n \geq m \geq N_2$. Then for all $n \geq m \geq \max(N_1, N_2)$ and all $x \in I$,

$$\begin{aligned}\left|\sum_{k=m}^{n}(-1)^n\left(\frac{x^2}{n^2}+\frac{1}{n}\right)\right| &\leq \left|\sum_{k=m}^{n}(-1)^n\frac{1}{n}\right| + \sum_{k=m}^{n}\frac{x^2}{n^2}\\ &\leq \left|\sum_{k=m}^{n}(-1)^n\frac{1}{n}\right| + \sum_{k=m}^{n}\frac{M^2}{n^2}\\ &< \varepsilon.\end{aligned}$$

This shows that (*) converges absolutely by Theorem 7.8. That the series does not converge absolutely is clear from the fact that $\sum 1/n$ diverges. □

Theorem 107. *[Exercise 7.7] Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined for all positive integers n by*

$$f_n(x) = \frac{x}{1+nx^2}.$$

Then $\{f_n\}$ converges uniformly to a function f, and the equation

$$(*)\qquad f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. Let $\varepsilon > 0$ be given and choose an integer N such that $N > 1/\varepsilon^2$. Let $n \geq N$ and $x \in \mathbb{R}$. If $|x| < \varepsilon$ then

$$\begin{aligned}\left|\frac{x}{1+nx^2}\right| &\leq |x| \\ &< \varepsilon.\end{aligned}$$

Otherwise,

$$\begin{aligned}\left|\frac{x}{1+nx^2}\right| &\leq \left|\frac{1}{nx}\right| \\ &< \varepsilon.\end{aligned}$$

This shows that $f_n \to 0$ uniformly on $\mathbb{R}$. For each n we have

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}.$$

If $x \neq 0$ then $f_n'(x) \to 0$ as $n \to \infty$ so that (*) is true, but $f_n'(0) = 1$ while $f'(0) = 0$, which contradicts (*). □

Theorem 108. *[Exercise 7.8] If*

$$I(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{otherwise,} \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, then the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$. Additionally, f is continuous for every $x \neq x_n$.

Proof. Applying Theorem 7.10 shows that the series converges uniformly on $[a, b]$ since

$$|c_n I(x - x_n)| \leq |c_n|$$

for each n and $\sum |c_n|$ converges. If $x \neq x_n$, then there exists a neighborhood N of x such that $N \cap \{x_n\}$ is empty. It is clear from the definition that f is constant on N, that is, $f(t) = f(u)$ for all $t, u \in N$. This shows that f is continuous at x. □

Theorem 109. *[Exercise 7.9] Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Then*

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$.

Proof. Let $\varepsilon > 0$ be given. Choose an integer N_1 such that $|f_n(t) - f(t)| < \varepsilon/2$ whenever $t \in E$ and $n \geq N_1$. By Theorem 7.12, f is continuous on E, so that we may choose a $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon/2$ whenever $|t - x| < \delta$, and choose an integer N_2 such that $|x_n - x| < \delta$ whenever $n \geq N_2$. Then for all $n \geq \max(N_1, N_2)$,

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \varepsilon. \end{aligned}$$

□

Theorem 110. *[Exercise 7.11] Let $\{f_n\}, \{g_n\}$ be sequences in a set E. If*

(1) *$\sum f_n$ has uniformly bounded partial sums,*
(2) *$g_n \to 0$ uniformly on E, and*
(3) *$g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$,*

then $\sum f_n g_n$ converges uniformly on E.

Proof. Note that $g_k(x) \geq 0$ for all $x \in E$ and $k \geq 1$, since each $\{g_n(x)\}$ is monotonic. Let $\varepsilon > 0$ be given. Since $\sum f_n$ has uniformly bounded partial sums, we can let $M = \sup_{x \in E} |A_n(x)|$ where $A_n(x)$ denotes the partial sums of $\sum f_n(x)$. Choose an integer N such that $g_N < \varepsilon/(2M)$. Then for all $n \geq m \geq N$ and $x \in E$,

$$\begin{aligned} \left| \sum_{k=m}^{n} f_n(x) g_n(x) \right| &= \left| \sum_{k=m}^{n-1} A_k(x)[g_k(x) - g_{k+1}(x)] + A_n(x) g_n(x) - A_{m-1}(x) g_m(x) \right| \\ &\leq \sum_{k=m}^{n-1} |A_k(x)| \, [g_k(x) - g_{k+1}(x)] + |A_n(x) g_n(x)| + |A_{m-1}(x) g_m(x)| \\ &\leq M \left(\sum_{k=m}^{n-1} [g_k(x) - g_{k+1}(x)] + g_n(x) + g_m(x) \right) \\ &= 2M g_m(x) \\ &< \varepsilon. \end{aligned}$$

□

Theorem 111. *Let $\{f_n\}$ be a sequence of functions that converge uniformly to f on $[a, \infty)$, where $\lim_{x \to \infty} f_n(x)$ exists for each n. Let*

$$A_n = \lim_{x \to \infty} f_n(x);$$

then $\{A_n\}$ converges, and

$$\lim_{x \to \infty} f(x) = \lim_{n \to \infty} A_n.$$

Proof. Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to f, there exists an integer N such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever every $x \geq a$ and $m, n \geq N$. By Corollary 30, $|A_n - A_m| < \varepsilon$ for all $m, n \geq N$. This shows that $\{A_n\}$ converges to some A. Choose an integer N such that $|f(x) - f_N(x)| < \varepsilon/3$ for all $x \geq a$ and $|A_N - A| < \varepsilon/3$. Then choose a M such that $|f_N(x) - A_N| < \varepsilon/3$ for all $x \geq M$, so that

$$\begin{aligned} |f(x) - A| &\leq |f(x) - f_N(x)| + |f_N(x) - A_N| + |A_N - A| \\ &< \varepsilon \end{aligned}$$

whenever $x \geq \max(a, M)$. This completes the proof. □

Theorem 112. *[Exercise 7.12] Let $g, f_n : (0, \infty) \to \mathbb{R}$ be functions Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$. If $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $[0, \infty)$, and*

$$\int_0^\infty g(x)\, dx < \infty,$$

then

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx,$$

provided that all improper integrals exist.

Proof. Define $F_n : [0, \infty) \to \mathbb{R}$ for each n and $F : [0, \infty) \to \mathbb{R}$ by

$$\begin{aligned} F_n(b) &= \int_0^b f_n(x)\, dx, \\ F(b) &= \int_0^b f(x)\, dx, \end{aligned}$$

and let $L = \int_0^\infty g(x)\, dx$ for convenience. For every b,

$$\lim_{n\to\infty} F_n(b) = F(b)$$

by Theorem 7.16, so that $F_n \to F$ pointwise on $[0, \infty)$. We also want to show that convergence is uniform. Let $\varepsilon > 0$ be given. Choose a $M \geq 0$ such that

$$\begin{aligned} \int_M^\infty g(x)\, dx &= L - \int_0^M g(x)\, dx \\ &< \frac{\varepsilon}{4}, \end{aligned}$$

and choose an integer N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2M}$$

whenever $n \geq N$ and $0 \leq x \leq M$. Then for all $b \geq M$ and $n \geq N$,

$$\begin{aligned} |F_n(b) - F(b)| &= \left| \int_0^b [f_n(x) - f(x)]\, dx \right| \\ &\leq \int_0^b |f_n(x) - f(x)|\ dx \\ &\leq \int_0^M |f_n(x) - f(x)|\ dx + 2\int_M^\infty g(x)\, dx \\ &< \varepsilon, \end{aligned}$$

while $|F_n(b) - F(b)| < \varepsilon/4 < \varepsilon$ trivially when $b < M$. The result then follows from applying Theorem 111 on $\{F_n\}$. □

Theorem 113. *[Exercise 7.13] Let $\{f_n\}$ be a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n.*

(1) *There is a function f and a sequence $\{n_k\}$ such that*

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}$.

(2) *If f is continuous, then $f_{n_k} \to f$ uniformly on compact sets.*

Proof. By Theorem 7.23, there exists a subsequence of functions $\{f_{n_k}\}$ such that $\{f_{n_k}(r)\}$ converges to some $f(r)$ for all $r \in \mathbb{Q}$. For all $x \in \mathbb{R}$, define

$$f(x) = \sup_{r \leq x, r\in\mathbb{Q}} f(r).$$

Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and suppose that f is continuous at x. Let $L = \lim_{k\to\infty} f_{n_k}(x)$; we want to show that $f(x) = L$. For every rational $r \leq x$ we have $f_{n_k}(r) \leq f_{n_k}(x)$ and therefore $f(r) \leq L$ by taking $k \to \infty$. This shows that $f(x) \leq L$. Suppose that $f(x) < L$, and choose a $\varepsilon > 0$ with $f(x) < f(x) + \varepsilon < L$. Choose a $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ whenever $|x - t| < \delta$. If $r \in \mathbb{Q}$ with $x < r < x + \delta$, then $f(r) < L$. But

$$L = \lim_{k\to\infty} f_{n_k}(x) \leq \lim_{k\to\infty} f_{n_k}(r) = f(r) < L,$$

which is a contradiction. Therefore $f(x) = L$. If $x < y$ then $f(x) = \lim_{k\to\infty} f_{n_k}(x) \leq \lim_{k\to\infty} f_{n_k}(y) = f(y)$; by Theorem 4.30, f has at most a countable number of discontinuities $\{t_i\}$. Applying Theorem 7.23 again to $\{t_i\}$ produces a subsequence $\left\{f_{n_j}\right\}$ of $\{f_{n_k}\}$ such that $f_{n_j}(t_i)$ converges to some u_i for every i. Redefining $f(x)$ using the new subsequence $\left\{f_{n_j}\right\}$ proves (1).

For (2), let f be a continuous function and let $\{n_k\}$ be a sequence such that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for every $x \in \mathbb{R}$. Let $E \subseteq \mathbb{R}$ be a compact set and let $\varepsilon > 0$ be given. By Theorem 4.19, f is uniformly continuous on E, so there exists a $\delta > 0$ such that

$|f(x) - f(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. Let $A = \inf E$ and $B = \sup E$; construct a set of points $\{x_1, \ldots, x_n\}$ where $A = x_1 \le \cdots \le x_n = B$ and $x_{i+1} - x_i < \delta/2$ for all $1 \le i \le n-1$. Then for each $1 \le i \le n-1$ we have

$$|f(x_{i+1}) - f(x_i)| = \left| \lim_{k\to\infty} [f_{n_k}(x_{i+1}) - f_{n_k}(x_i)] \right| < \varepsilon/3$$

and we may choose an integer N_i such that both $|f_{n_k}(x_{i+1}) - f_{n_k}(x_i)| < \varepsilon/3$ and $|f_{n_k}(x_i) - f(x_i)| < \varepsilon/3$ whenever $k \ge N_i$; let $N = \max\{N_i\}$. Let $x \in E$ and choose a j such that $x \in [x_j, x_{j+1}]$. Then for all $k \ge N$, since each f_n is monotonically increasing we have

$$\begin{aligned} 0 \le f_{n_k}(x) - f_{n_k}(x_j) &\le f_{n_k}(x_{j+1}) - f_{n_k}(x) \\ &< \varepsilon/3 \end{aligned}$$

so that

$$\begin{aligned} |f_{n_k}(x) - f(x)| &\le |f_{n_k}(x) - f_{n_k}(x_j)| + |f_{n_k}(x_j) - f(x_j)| + |f(x_j) - f(x)| \\ &< \varepsilon. \end{aligned}$$

This completes the proof. □

Theorem 114. *[Exercise 7.15] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$. If $\{f_n\}$ is equicontinuous on $[0, 1]$, then f is constant on $[0, \infty)$.*

Proof. Suppose that f is not constant and without loss of generality, let $0 \le x_1 < x_2$ with $f(x_1) < f(x_2)$. Since $\{f_n\}$ is equicontinuous, there exists a $\delta > 0$ such that $|f(nt) - f(nu)| < [f(x_2) - f(x_1)]/2$ whenever $n \ge 1$, $0 \le t, u \le 1$, and $|t - u| < \delta$. Let n be an integer with

$$n > \max\left\{ \frac{x_2 - x_1}{\delta}, x_1, x_2 \right\}.$$

Set $t = x_2/n$ and $u = x_1/n$; then $0 \le t, u < 1$ and $|t - u| = (x_2 - x_1)/n < \delta$ so that

$$\begin{aligned} |f(nt) - f(nu)| &= f(x_2) - f(x_1) \\ &< [f(x_2) - f(x_1)]/2, \end{aligned}$$

which is a contradiction. □

Theorem 115. *[Exercise 7.16] Let $\{f_n\}$ be an equicontinuous sequence of functions on a compact set K. If $\{f_n\}$ converges pointwise on K, then $\{f_n\}$ converges uniformly on K.*

Proof. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $n \ge 1$, $x, y \in K$, and $|x - y| < \delta$. The proof is now almost identical to part (2) of Theorem 113. □

Theorem 116. *[Exercise 7.18] Let* $\{f_n\}$ *be a uniformly bounded sequence of functions which are Riemann-integrable on* $[a, b]$*, and let*

$$F_n(x) = \int_a^x f_n(t)\,dt$$

for $a \le x \le b$*. Then there exists a subsequence* $\{F_{n_k}\}$ *which converges uniformly on* $[a, b]$*.*

Proof. Since $\{f_n\}$ is uniformly bounded, there exists a $M > 0$ such that $|f_n(t)| < M$ for all n and t. Let $\varepsilon > 0$ be given. Then for all $|x - y| < \varepsilon/M$ and all n we have

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left| \int_y^x f_n(t)\,dt \right| \\ &\le \int_y^x |f_n(t)|\,dt \\ &\le M\,|x - y| \\ &< \varepsilon, \end{aligned}$$

which shows that $\{F_n\}$ is equicontinuous. Clearly, $\{F_n\}$ is also uniformly bounded. The result follows from Theorem 7.25. □

Theorem 117. *[Exercise 7.20] If* f *is continuous on* $[0, 1]$ *and if*

$$\int_0^1 f(x)x^n\,dx = 0$$

for all $n = 0, 1, 2, \ldots,$ *then* $f(x) = 0$ *on* $[0, 1]$*.*

Proof. By Theorem 7.26, there exists a sequence of polynomials P_n such that $P_n \to f$ uniformly on $[0, 1]$. For each n, write $P_n(x) = \sum_k a_k x^k$ so that

$$\begin{aligned} \int_0^1 f(x)P_n(x)\,dx &= \int_0^1 f(x) \sum_k a_k x^k\,dx \\ &= \sum_k a_k \int_0^1 f(x)x^k\,dx \\ &= 0. \end{aligned}$$

Since f is bounded on $[0,1]$, $fP_n \to f^2$ uniformly on $[0,1]$ by Theorem 102 and

$$\begin{aligned} f(x) &= \lim_{n\to\infty} P_n(x) \\ \int_0^1 f(x)^2\,dx &= \int_0^1 \lim_{n\to\infty} f(x)P_n(x)\,dx \\ &= \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx \\ &= 0. \end{aligned}$$

Therefore $f(x)^2 = 0$ on $[0,1]$. $\square$

Theorem 118. *[Exercise 7.23] Let $P_0 = 0$, and define, for $n = 0, 1, 2, \ldots$,*

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Then

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-1,1]$.

Proof. We have the identity

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + \frac{[|x| + P_n(x)]\,[|x| - P_n(x)]}{2} \\ |x| - P_{n+1}(x) &= |x| - P_n(x) - \frac{[|x| + P_n(x)]\,[|x| - P_n(x)]}{2} \\ &= [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right]. \end{aligned}$$

By induction on n we have $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ for all n whenever $|x| \le 1$. By iteration,

$$\begin{aligned} |x| - P_n(x) &= |x| \prod_{k=0}^{n-1} \left(1 - \frac{|x| + P_k(x)}{2}\right) \\ &\le |x| \prod_{k=0}^{n-1} \left(1 - \frac{|x|}{2}\right) \\ &= |x| \left(1 - \frac{|x|}{2}\right)^n. \end{aligned}$$

For $n \geq 1$, function $f(x) = x(1 - x/2)^n$ has derivative

$$\begin{aligned} f'(x) &= \left(1 - \frac{x}{2}\right)^n - \frac{nx}{2}\left(1 - \frac{x}{2}\right)^{n-1} \\ &= \left(1 - \frac{x}{2}\right)^{n-1}\left[1 - \left(\frac{n+1}{2}\right)x\right] \end{aligned}$$

which vanishes at $x_0 = 2/(n+1)$. This value satisfies $f(x_0) \leq x_0$. Since $f'(x) > 0$ when $0 \leq x < x_0$ and $f'(x) < 0$ when $x_0 < x \leq 1$,

$$|x| - P_n(x) \leq \frac{2}{n+1}$$

for all $|x| \leq 1$. The result follows taking n large enough. □