Chapter 2. Topological Spaces

Example 1. [Exercise 2.2] Show that each of the following is a topological space.

(1) Let X denote the set $\{1, 2, 3\}$, and declare the open sets to be $\{1\}$, $\{2, 3\}$, $\{1, 2, 3\}$, and the empty set.
(2) Any set X whatsoever, with $\mathcal{T} = \{\text{all subsets of } X\}$. This is called the **discrete topology** on X, and $(X, \mathcal{T})$ is called a **discrete space**.
(3) Any set X, with $\mathcal{T} = \{\emptyset, X\}$. This is called the **trivial topology** on X.
(4) Any metric space (M, d), with $\mathcal{T}$ equal to the collection of all subsets of M that are open in the metric space sense. This topology is called the **metric topology** on M.

We only check (4). It is clear that $\emptyset$ and M are in $\mathcal{T}$. Let $U_1, \ldots, U_n \in \mathcal{T}$ and let $p \in U_1 \cap \cdots \cap U_n$. There exist open balls $B_1 \subseteq U_1, \ldots, B_n \subseteq U_n$ that contain p, so the intersection $B_1 \cap \cdots \cap B_n \subseteq U_1 \cap \cdots \cap U_n$ is an open ball containing p. This shows that $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in A}$ be some collection of elements in $\mathcal{T}$ which we can assume to be nonempty. If $p \in \bigcup_{\alpha \in A} U_\alpha$ we can choose some $\alpha \in A$, and there exists an open ball $B \subseteq U_\alpha$ containing p. But $B \subseteq \bigcup_{\alpha \in A} U_\alpha$, so this shows that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Theorem 2. *[Exercise 2.9] Let X be a topological space and let $A \subseteq X$ be any subset.*

(1) *A point q is in the interior of A if and only if q has a neighborhood contained in A.*
(2) *A point q is in the exterior of A if and only if q has a neighborhood contained in $X \setminus A$.*
(3) *A point q is in the boundary of A if and only if every neighborhood of q contains both a point of A and a point of $X \setminus A$.*
(4) $\operatorname{Int} A$ *and* $\operatorname{Ext} A$ *are open in X, while ∂A is closed in X.*
(5) *A is open if and only if $A = \operatorname{Int} A$, and A is closed if and only if $A = \overline{A}$.*
(6) *A is closed if and only if it contains all its boundary points, which is true if and only if $A = \operatorname{Int} A \cup \partial A$.*
(7) $\overline{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$.

Proof. (1) and (2) are obvious. Let $q \in \partial A$ and let N be a neighborhood of q. By (1) and (2), N is not contained in A or $X \setminus A$, i.e. N contains a point of A and a point of $X \setminus A$. Conversely, if every neighborhood of q contains both a point of A and a point of $X \setminus A$ then $q \notin (\operatorname{Int} A \cup \operatorname{Ext} A)$ by (1) and (2). Parts (4) and (5) are obvious. If A is closed and $p \in \partial A$ then

$$p \in X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A) = X \setminus (\operatorname{Int} A \cup (X \setminus A)) \subseteq A.$$

If A contains all its boundary points then $\operatorname{Int} A \cup \partial A \subseteq A$, so $A = \operatorname{Int} A \cup \partial A$ since it is always true that $A \subseteq \operatorname{Int} A \cup \partial A$. If $A = \operatorname{Int} A \cup \partial A$ then A is closed since

$A = \operatorname{Int} A \cup (X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)) = X \setminus \operatorname{Ext} A$ and $\operatorname{Ext} A$ is open. This proves (6). For (7),

$$\operatorname{Int} A \cup \partial A = \operatorname{Int} A \cup (X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)) = X \setminus \operatorname{Ext} A = \overline{A}$$

and $A \cup \partial A = \operatorname{Int} A \cup \partial A$ since $\operatorname{Int} A \cup \partial A \subseteq A \cup \partial A$ and $p \in A \cup \partial A$ implies that $p \in \partial A$ or $p \in A \setminus \partial A = \operatorname{Int} A$. □

Theorem 3. *[Exercise 2.10] A set A in a topological space X is closed if and only if it contains all of its limit points.*

Proof. Suppose A is closed and let $q \in X \setminus A$ be a limit point of A. Since every neighborhood of q contains both a point of $X \setminus A$ and a point of A, we have $q \in \partial A \subseteq \overline{A}$. Conversely, suppose that A contains all of its limit points. If $q \in \partial A \setminus A$ and N is a neighborhood of q then N contains a point of A which cannot be equal to q since $q \notin A$. Therefore q is a limit point of A, and is contained in A. This shows that A contains all its boundary points, i.e. A is closed. □

Theorem 4. *[Exercise 2.11] A subset $A \subseteq X$ is dense if and only if every nonempty open set in X contains a point of A.*

Proof. Suppose that $\overline{A} = X$. Let U be a nonempty open set in X and let p be any point in U. Since $\operatorname{Ext} A$ is empty, no neighborhood of p is contained in $X \setminus A$. But U is a neighborhood of p, so U contains a point of A. Conversely, suppose that every nonempty open set in X contains a point of A. Then every $p \in X \setminus A$ is a limit point of A, so $p \in \overline{A}$ and therefore $X \subseteq \overline{A}$. □

Theorem 5. *[Exercise 2.12] Let (M, d) be a metric space with the usual topology. The following are equivalent for a sequence $\{q_i\}$ and a point q in M:*

(1) *For every neighborhood U of q there exists an integer N such that $q_i \in U$ for all $i \geq N$.*
(2) *For every $\varepsilon > 0$ there exists an integer N such that $d(q_i, q) < \varepsilon$ for all $i \geq N$.*

Proof. The direction (1) $\Rightarrow$ (2) is clear. Suppose (2) holds and let U be a neighborhood of q. Since U is open, there exists an open ball $B \subseteq U$ of radius r around q, so there exists an integer N such that $d(q_i, q) < \varepsilon$ for all $i \geq N$. But then $q_i \in B \subseteq U$ for all $i \geq N$, which proves (1). □

Theorem 6. *[Exercise 2.13] Let X be a discrete topological space. Then the only convergent sequences in X are the ones that are "eventually constant", that is, sequences $\{q_i\}$ such that $q_i = q$ for all i greater than some N.*

Proof. Suppose that $q_i \to q$ in X. Since $\{q\}$ is open, there exists an integer N such that $q_i \in \{q\}$, i.e. $q_i = q$, for all $i \geq N$. □

Theorem 7. *[Exercise 2.16] A map $f : X \to Y$ between topological spaces is continuous if and only if the inverse image of every closed set is closed.*

Proof. Suppose that f is continuous and let $A \subseteq Y$ be a closed set. Since $Y \setminus A$ is open, $f^{-1}(Y \setminus A)$ is open and therefore $f^{-1}(A) = X \setminus f^{-1}(Y \setminus A)$ is closed. The converse is similar. □

Theorem 8. *[Exercise 2.18] Let X, Y, and Z be topological spaces.*

1. *Any constant map $f : X \to Y$ is continuous.*
2. *The identity map $\mathrm{Id} : X \to X$ is continuous.*
3. *If $f : X \to Y$ is continuous, so is the restriction of f to any open subset of X.*
4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f : X \to Z$.*

Proof. For (1), suppose that $f(x) = y$ for all $x \in X$. Let $U \subseteq Y$ be an open set. If $y \in U$ then $f^{-1}(U) = X$ and if $y \notin U$ then $f^{-1}(U) = \emptyset$; in either case, $f^{-1}(U)$ is open. Part (2) is obvious. Part (3) follows from the fact that the topology is closed under finite intersections. For (4), let $U \subseteq Z$ be an open set. Since g is continuous $g^{-1}(U)$ is open, and since f is continuous $f^{-1}(g^{-1}(U))$ is open. But $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$, so this proves that $g \circ f$ is continuous. □

Theorem 9. *[Exercise 2.20] "Homeomorphic" is an equivalence relation.*

Proof. For any space X the identity map $\mathrm{Id} : X \to X$ is a homeomorphism between X and itself. If $\varphi : X \to Y$ is a homeomorphism then $\varphi^{-1} : Y \to X$ is also a homeomorphism. Finally, if $\varphi : X \to Y$ and $\psi : Y \to Z$ are homeomorphisms then $\psi \circ \varphi : X \to Z$ is a homeomorphism since $(\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1}$ is continuous. □

Theorem 10. *[Exercise 2.21] Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $f : X_1 \to X_2$ be a bijective map. Then f is a homeomorphism if and only if $f(\mathcal{T}_1) = \mathcal{T}_2$ in the sense that $U \in \mathcal{T}_1$ if and only if $f(U) \in \mathcal{T}_2$.*

Proof. This is clear from the definition. □

Theorem 11. *[Exercise 2.27] Let $C = \{(x, y, z) \mid \max(|x|, |y|, |z|) = 1\}$ and let $\varphi : C \to \mathbb{S}^2$ be given by*

$$\varphi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

Then φ is a homeomorphism with inverse

$$\varphi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Proof. Note that φ and φ^{-1} are continuous; it suffices to check

$$\varphi(\varphi^{-1}(x,y,z)) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x,y,z)$$

since $(x,y,z) \in \mathbb{S}^2$ implies that $\sqrt{x^2+y^2+z^2} = 1$ and

$$\varphi^{-1}(\varphi(x,y,z)) = \frac{(x,y,z)}{\max(|x|,|y|,|z|)} = (x,y,z)$$

since $(x,y,z) \in C$. □

Theorem 12. *[Exercise 2.28] Let X denote the half-open interval $[0,1) \subseteq \mathbb{R}$, and let S^1 denote the unit circle in $\mathbb{R}^2$. Define a map $a : X \to S^1$ by $a(t) = (\cos 2\pi t, \sin 2\pi t)$. Then a is continuous and bijective but not a homeomorphism.*

Proof. Let $I = [0, 1/2)$, which is open in X. Since $a(I)$ is not open in S^1, the inverse of a cannot be continuous. □

Theorem 13. *[Exercise 2.31]*

(1) *Every local homeomorphism is an open map.*
(2) *Every homeomorphism is a local homeomorphism.*
(3) *Every bijective continuous open map is a homeomorphism.*
(4) *Every bijective local homeomorphism is a homeomorphism.*

Proof. Let $f : X \to Y$ be a local homeomorphism and let $U \subseteq X$ be an open set. Let $y \in f(U)$ so that $y = f(x)$ for some $x \in U$. There exists a neighborhood V of x such that $f(V)$ is open and $f|_V : V \to f(V)$ is a homeomorphism; in particular, $N_y = f(V \cap U) \subseteq f(U)$ is open. Therefore $f(U) = \bigcup_{y \in f(U)} N_y$ is open, which proves (1). Parts (2) and (3) are obvious, and (4) follows from (1) and (3). □

Theorem 14. *[Exercise 2.33] Let Y be a trivial topological space (that is, a set with the trivial topology). Then every sequence Y converges to every point of Y.*

Proof. Let $\{r_i\}$ be a sequence in Y and let r be any element of Y. If U is a neighborhood of r then $U = Y$, so it is trivial that $r_i \to r$. □

Theorem 15. *[Exercise 2.35] Suppose X is a topological space, and for every $p \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \{p\}$. Then X is Hausdorff.*

Proof. Let p, q be distinct points of X and let $f : X \to \mathbb{R}$ be a continuous function such that $f^{-1}(\{0\}) = \{p\}$. Then $f(q) \neq 0$, and we can choose disjoint neighborhoods U of 0 and V of $f(q)$ in $\mathbb{R}$. We have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, while $f^{-1}(U)$ is a neighborhood of p and $f^{-1}(V)$ is a neighborhood of q. □

Theorem 16. *[Exercise 2.38] The only Hausdorff topology on a finite set is the discrete topology.*

Proof. Let X be a finite set with a Hausdorff topology $\mathcal{T}$. By Proposition 2.37, every one-point set in X is closed. But every subset of X can be written as a finite union of one-point sets, so every subset of X is closed and $\mathcal{T}$ must be the discrete topology. □

Example 17. In each of the following cases, the given set $\mathcal{B}$ is a basis for the given topology.

(1) M is a metric space with the metric topology, and $\mathcal{B}$ is the collection of all open balls in M.
(2) X is a set with the discrete topology, and $\mathcal{B}$ is the collection of all one-point subsets of X.
(3) X is a set with the trivial topology, and $\mathcal{B} = \{X\}$.

(1) follows trivially from the definition of the metric topology. For (2), let U be an open set in X. If $x \in U$ then $x \in \{x\} \subseteq U$, so $\mathcal{B}$ is a basis. For (3), if U is an open set then it is either empty or equal to X. In the latter case, for any $x \in U$ we have $x \in X \subseteq U$, so $\mathcal{B}$ is a basis.

Theorem 18. *[Exercise 2.42] The following collections are bases for the Euclidean topology on $\mathbb{R}^n$:*

(1) $\mathcal{B}_1 = \{C_s(x) : x \in \mathbb{R}^n$ *and* $s > 0\}$*, where* $C_s(x)$ *is the open cube of side* s *around* x*:*
$$C_s(x) = \{y = (y_1, \ldots, y_n) : |x_i - y_i| < s/2,\ i = 1, \ldots, n\}$$
(2) $\mathcal{B}_2 = \{B_r(x) : r$ *is rational and* x *has rational coordinates*$\}$.

Proof. Let U be an open set and let $x \in U$. There exists some open ball $B \subseteq U$ of radius r around x. For (1), the open cube $C_{r/\sqrt{2}}(x)$ is an element of $\mathcal{B}_1$ that is contained in B. For (2), choose any rational r' with $0 < r' \leq r/2$ and any point $x' \in B_{r'}(x)$ with rational coordinates; then $B_{r'}(x') \subseteq B$ is an element of $\mathcal{B}_2$ that contains x. □

Theorem 19. *[Exercise 2.51]*

(1) *Every second countable space has a countable dense subset.*
(2) *A metric space is second countable if and only if it has a countable dense subset.*

Proof. Let X be a second countable space and let $\mathcal{B}$ be a countable basis for X. Let E be the countable set formed by choosing a point from each $B \in \mathcal{B}$. Let U be a nonempty open set in X and let $x \in U$. There exists a $B \in \mathcal{B}$ with $x \in B$ and $B \subseteq U$, so U contains a point from E. This shows that E is a countable dense subset of X.

Now let (M, d) be a metric space with a countable dense subset E. We want to show that the countable set

$$\mathcal{B} = \{B_r(p) : (p, r) \in E \times \mathbb{Q}\}$$

is a basis for M. Let U be an open set in M, let $x \in U$, and let $B \subseteq U$ be an open ball of radius r containing x. Since E is dense in M, there exists a point $p \in E \cap B_{r/2}(x)$. Let r' be a rational number with $d(p, x) < r' < r/2$. Then $B_{r'}(p)$ is an element of $\mathcal{B}$ with $x \in B_{r'}(p)$ and $B_{r'}(p) \subseteq B$, since

$$d(x, y) \leq d(x, p) + d(p, y) < r$$

for every $y \in B_{r'}(p)$. □

Theorem 20. *[Exercise 2.54] A topological space X is a 0-manifold if and only if it is a countable discrete space.*

Proof. Let $x \in X$ and let N be a neighborhood of x homeomorphic to an open subset of $\mathbb{R}^0$, i.e. a point. Then N must contain exactly one element, so $N = \{x\}$. This shows that every one-point subset of X is open, and therefore X is discrete. Also, any second countable discrete space must also be countable, so X is a countable discrete space. □

Lemma 21. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. If $A \subseteq X$ and $B \subseteq Y$ are subsets with $f(A) \subseteq B$, then $f(\overline{A}) \subseteq \overline{B}$.*

Proof. Suppose that $f(x) \in \operatorname{Ext} B$ for some $x \in \overline{A}$. Choose a neighborhood U of $f(x)$ such that $U \subseteq Y \setminus B$. Then $f^{-1}(U)$ is a neighborhood of x such that $f^{-1}(U) \subseteq X \setminus A$, which contradicts the fact that $x \in \overline{A}$. □

Lemma 22. *Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ be topological spaces with the same underlying set. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Then $\mathcal{T}_1 = \mathcal{T}_2$ if and only if $\mathcal{B}_2 \subseteq \mathcal{T}_1$ and $\mathcal{B}_1 \subseteq \mathcal{T}_2$.*

Proof. If $\mathcal{T}_1 = \mathcal{T}_2$ then it is clear that $\mathcal{B}_2 \subseteq \mathcal{T}_2 = \mathcal{T}_1$ and $\mathcal{B}_1 \subseteq \mathcal{T}_1 = \mathcal{T}_2$. Conversely, suppose that $\mathcal{B}_2 \subseteq \mathcal{T}_1$ and $\mathcal{B}_1 \subseteq \mathcal{T}_2$. Every open set in $\mathcal{T}_1$ can be written as the union of elements from $\mathcal{B}_1$, which are open in $\mathcal{T}_2$ by assumption. Therefore every open set in $\mathcal{T}_1$ is also open in $\mathcal{T}_2$. Similarly, every open set in $\mathcal{T}_2$ is also open in $\mathcal{T}_1$. □

Lemma 23. *Let (M_1, d_1) and (M_2, d_2) be metric spaces with $M = M_1 = M_2$ (but different metrics). Then the following are equivalent:*

1. *The two spaces have the same topology.*
2. *Let $x \in M$. Every open ball in M_1 around x contains an open ball in M_2 around x, and vice versa.*

Proof. Suppose that the two spaces have the same topology. If B is an open ball in M_1 around $x \in M$ then B is open in M_2, so there is an open ball $B' \subseteq B$ in M_2 around x. An identical argument holds when B is an open ball in M_2. Now suppose that (2) holds and let U be open in M_1. If $x \in U$ then there exists an open ball $B \subseteq U$ in M_1 around x, so there is an open ball $B' \subseteq B$ in M_2 around x. This shows that U is open in M_2. Similarly, every set that is open in M_2 is also open in M_1. This shows that the two spaces have the same topology. □

Example 24. [Problem 2-1] Let X be an infinite set. Consider the following collections of subsets of X:

$$\mathcal{T}_1 = \{U \subseteq X : X \setminus U \text{ is finite or is all of } X\};$$
$$\mathcal{T}_2 = \{U \subseteq X : X \setminus U \text{ is infinite or is empty}\};$$
$$\mathcal{T}_3 = \{U \subseteq X : X \setminus U \text{ is countable or is all of } X\}.$$

For each collection, determine whether it is a topology.

$\mathcal{T}_1$ is a topology; $\mathcal{T}_2$ is not a topology; $\mathcal{T}_3$ is a topology.

Theorem 25. *[Problem 2-3] Let X be a topological space and let B be a subset of X.*

(1) $\overline{X \setminus B} = X \setminus \operatorname{Int} B$.
(2) $\operatorname{Int}(X \setminus B) = X \setminus \overline{B}$.

Proof. Suppose that $x \in \overline{X \setminus B}$. We can assume that $x \in B$, for otherwise $x \in X \setminus B \subseteq X \setminus \operatorname{Int} B$. Then x is a limit point of $X \setminus B$, so any neighborhood of x contains a point of $X \setminus B$. This shows that $x \notin \operatorname{Int} B$, i.e. $x \in X \setminus \operatorname{Int} B$. Conversely, suppose that $x \notin \overline{X \setminus B}$. Then there is a neighborhood N of x that does not contain a point of $X \setminus B$. Then $N \subseteq B$, which shows that $x \in \operatorname{Int} B$. This proves (1). For (2), apply (1) with $X \setminus B$ to get $\overline{B} = X \setminus \operatorname{Int}(X \setminus B)$ and take complements to get $X \setminus \overline{B} = \operatorname{Int}(X \setminus B)$. □

Example 26. [Problem 2-4] Let $X = \{1, 2, 3\}$. Give a list of topologies on X such that any topology on X is homeomorphic to exactly one on your list.

The topologies up to homeomorphism are:

$$\mathcal{T}_1 = \{\emptyset, \{1,2,3\}\}$$
$$\mathcal{T}_2 = \{\emptyset, \{1\}, \{1,2,3\}\}$$
$$\mathcal{T}_3 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$$
$$\mathcal{T}_4 = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$$
$$\mathcal{T}_5 = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$$
$$\mathcal{T}_6 = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$$
$$\mathcal{T}_7 = \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{1,2,3\}\}$$

$$\mathcal{T}_8 = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$$
$$\mathcal{T}_9 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Theorem 27. *[Problem 2-7] Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighborhood of p contains infinitely many points of A.*

Proof. Construct a sequence of distinct points $\{q_n\}$ in A as follows. Choose any point $q_1 \in A$ not equal to p. Suppose we have chosen the distinct points $q_1, \ldots, q_n$, none of which are equal to p. For each i, there exist neighborhoods U_i of q_i and V_i of p such that $U_i \cap V_i = \emptyset$. Then $V = V_1 \cap \cdots \cap V_n$ is a neighborhood of p, so V contains some point $q_{n+1} \in A$ not equal to p. Furthermore, q_{n+1} is not equal to any q_i for $i \leq n$ since $V \cap U_i = \emptyset$ for every i. □

Theorem 28. *[Problem 2-8] Let X be a Hausdorff space, let $A \subseteq X$, and let A' denote the set of limit points of A. Then A' is closed in X.*

Proof. Let x be a limit point of A' and let E be a neighborhood of x. Then there exists some $y \in E \cap A'$ not equal to x. Choose neighborhoods U of x and V of y such that $U \cap V = \emptyset$ and $U, V \subseteq E$. Since y is a limit point of A', there exists some $z \in V \cap A$ not equal to y. Then $z \neq x$ since U and V are disjoint, which proves that $x \in A'$. Therefore A' is closed in X. □

Theorem 29. *[Problem 2-9] Let X be a discrete space, Y be a space with the trivial topology, and Z be any topological space. Any maps $f : X \to Z$ and $g : Z \to Y$ are continuous, and if Z is Hausdorff, then the only continuous maps $h : Y \to Z$ are constant maps.*

Proof. Suppose that Z is Hausdorff and $h : Y \to Z$ is non-constant continuous map. Choose distinct points $z_1, z_2 \in Z$ such that $z_1 = h(y_1)$ and $z_2 = h(y_2)$ for some $y_1, y_2 \in Y$, and choose neighborhoods U_1 of z_1 and U_2 of z_2 with $U_1 \cap U_2 = \emptyset$. The set $h^{-1}(U_1)$ is nonempty and open in Y, so $h^{-1}(U_1) = Y$. But then $z_2 \in h(Y) = h(h^{-1}(U_1)) = U_1$, which contradicts the fact that $U_1 \cap U_2 = \emptyset$. □

Theorem 30. *[Problem 2-11] Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{B}$ be a basis for the topology of X. Let $f(\mathcal{B})$ denote the collection $\{f(B) : B \in \mathcal{B}\}$ of subsets of Y. If f is surjective and open, then $f(\mathcal{B})$ is a basis for the topology of Y.*

Proof. Every set in $f(\mathcal{B})$ is open since f is an open map. Let U be an open set in Y and let $y \in U$. Since f is surjective we have $y = f(x)$ for some x, so $f^{-1}(U)$ is a neighborhood of x. There exists some $B \in \mathcal{B}$ with $x \in B$ and $B \subseteq f^{-1}(U)$, so $f(B) \in f(\mathcal{B})$ with $y \in f(B)$ and $f(B) \subseteq f(f^{-1}(U)) = U$. □

Theorem 31. *[Problem 2-12] Suppose X is a set, and $\mathcal{B}$ is any collection of subsets of X whose union equals X. Let $\mathcal{T}$ be the collection of all unions of finite intersections of elements of $\mathcal{B}$.*

(1) *$\mathcal{T}$ is a topology. (It is called the **topology generated by** $\mathcal{B}$, and $\mathcal{B}$ is called a **subbasis** for $\mathcal{T}$.)*
(2) *$\mathcal{T}$ is the "smallest" topology for which all the sets in $\mathcal{B}$ are open. More precisely, $\mathcal{T}$ is the intersection of all topologies containing $\mathcal{B}$.*

Proof. It is clear that $\emptyset, X \in \mathcal{T}$, and that arbitrary unions of open sets are also open sets. If

$$\bigcup_{\alpha\in A}(U_{\alpha,1}\cap\cdots\cap U_{\alpha,m_\alpha}) \quad \text{and} \quad \bigcup_{\beta\in B}(U_{\beta,1}\cap\cdots\cap U_{\beta,n_\beta})$$

are elements of $\mathcal{T}$, then their intersection is

$$\bigcup_{(\alpha,\beta)\in A\times B} U_{\alpha,1}\cap\cdots\cap U_{\alpha,m_\alpha}\cap U_{\beta,1}\cap\cdots\cap U_{\beta,n_\beta},$$

which is also in $\mathcal{T}$. This proves that $\mathcal{T}$ is a topology. To prove (2), we only need to show that for every topology $\mathcal{T}'$ containing $\mathcal{B}$, we have $\mathcal{T}\subseteq\mathcal{T}'$. But this is clear from the fact that $\mathcal{T}'$ is closed under arbitrary unions and finite intersections. □

Theorem 32. *[Problem 2-13] Let X be a totally ordered set with at least two elements. For any $a\in X$, define sets $L(a), R(a)\subseteq X$ by*

$$L(a) = \{c\in X : c < a\},$$
$$R(a) = \{c\in X : c > a\}.$$

*Give X the topology generated by the subbasis $\{L(a), R(a) : a\in X\}$, called the **order topology**.*

(1) *Each set of the form (a,b) is open in X and each set of the form $[a,b]$ is closed.*
(2) *X is Hausdorff.*
(3) *For any $a,b\in X$, $\overline{(a,b)} = [a,b]$.*

Proof. We can write $(a,b) = R(a)\cap L(b)$ and $[a,b] = (X\setminus L(a))\cap(X\setminus R(b))$, which shows that (a,b) is open and $[a,b]$ is closed. For (2), let a and b be distinct points in X. If there exists some c with $a<c<b$ then we have the neighborhoods $L(c)$ of a and $R(c)$ of b with $L(c)\cap R(c)=\emptyset$. Otherwise, the neighborhoods $L(b)$ of a and $R(a)$ of b satisfy $L(b)\cap R(a)=\emptyset$. This shows that X is Hausdorff. For (3), if E is a closed set containing (a,b) then $[a,b]\subseteq E$ since a and b are boundary points of (a,b). But $[a,b]$ is closed, so $\overline{(a,b)} = [a,b]$. □

Theorem 33. *[Problem 2-15] Let X be a first countable space.*

(1) *For any set* $A \subseteq X$ *and any point* $p \in X$*, we have* $p \in \overline{A}$ *if and only if there is a sequence* $\{p_n\}_{n=1}^{\infty}$ *in* A *such that* $p_n \to p$*.*
(2) *For any space* Y*, a map* $f : X \to Y$ *is continuous if and only if* f *takes convergent sequences in* X *to convergent sequences in* Y*.*

Proof. For (1), if $p \in A$ then the constant sequence $p_n = p$ converges to p. Otherwise, $p \notin A$ and p is a limit point of A. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable neighborhood basis for p and let $E_n = B_1 \cap \cdots \cap B_n$. For each n the set E_n is a neighborhood of p, so we can choose a point $p_n \in E_n \cap A$ since p is a limit point of A. We want to show that the sequence $\{p_n\}$ converges to p. Let U be a neighborhood of p. There exists a $B_N \in \mathcal{B}$ with $B_N \subseteq U$, and for all $n \geq N$ we have $p_n \in U$ since

$$p_n \in E_n \subseteq B_1 \cap \cdots \cap B_n \subseteq B_1 \cap \cdots \cap B_N.$$

Conversely, suppose that there is a sequence $\{p_n\}$ in A with $p_n \to p$. We may assume that $p_n \neq p$ for all n, for otherwise $p \in A$ and we are done. If U is a neighborhood of p then there exists an integer N such that $p_n \in U$ for all $n \geq N$. In particular, $p_N \in U \cap A$ and $p_N \neq p$ by our previous assumption, which proves that p is a limit point of A. □

Theorem 34. *[Problem 2-16] If* X *is a second countable topological space, then every collection of disjoint open subsets of* X *is countable.*

Proof. Let $\mathcal{B}$ be a countable basis for X and let $\{U_\alpha\}_{\alpha \in A}$ be a collection of disjoint nonempty open subsets. For each α, choose a point $x_\alpha \in U_\alpha$; there exists an element $B_\alpha \in \mathcal{B}$ with $x_\alpha \in B_\alpha$ and $B_\alpha \subseteq U_\alpha$. Since $B_\alpha \neq B_\beta$ whenever $\alpha \neq \beta$, we have a bijection between A and the countable set $\{B_\alpha\}$. □

Lemma 35. *Every metric space* M *is first countable.*

Proof. Let $p \in M$ and consider the countable collection of open balls $\left\{B_{1/n}(p) : n \in \mathbb{Z}^+\right\}$. □

Theorem 36. *Let* X *and* Y *be topological spaces with the property that every point in* X *has a neighborhood homeomorphic to an open set in* Y*. Then* X *is first countable if* Y *is first countable.*

Proof. Let $x \in X$. There exists a neighborhood N of x with a homeomorphism $f : N \to f(N)$, and there exists a countable neighborhood basis $\mathcal{B}$ of $f(x)$. If U is a neighborhood of x then $f(U \cap N)$ is a neighborhood of $f(x)$, so there exists a $B \in \mathcal{B}$ with $B \subseteq f(U \cap N)$. Then $f^{-1}(B) \subseteq U \cap N$, which shows that $\{f^{-1}(B) : B \in \mathcal{B}\}$ is a countable neighborhood basis of x. □

Corollary 37. *[Problem 2-21] All locally Euclidean spaces and metric spaces are first countable.*

Example 38. [Problem 2-22] Let $X = \mathbb{R}^2$ as a set, but with the topology determined by the following basis:

$$\mathcal{B} = \{\text{sets of the form } \{(c, y) : a < y < b\}\,, \text{ for fixed } a, b, c \in \mathbb{R}\}\,.$$

Determine which (if either) of the identity maps $X \to \mathbb{R}^2$, $\mathbb{R}^2 \to X$ is continuous.

The map $X \to \mathbb{R}^2$ is not continuous because the unit open ball B is not open in X. In particular, $(1, 0) \in B$ but there is no element of $\mathcal{B}$ that both contains $(1, 0)$ and is a subset of B. For if $U = \{(1, y) : a < y < b\} \in \mathcal{B}$ contains $(1, 0)$ then there are no values of a, b for which U is a subset of B. The map $\mathbb{R}^2 \to X$ is not continuous either because no set of the form $\{(c, y) : a < y < b\}$ is open in the usual metric topology on $\mathbb{R}^2$.

Theorem 39. *[Problem 2-23] Any manifold has a basis of coordinate balls.*

Proof. Suppose that $(M, \mathcal{T})$ is an n-manifold where $\mathcal{T} = \{U_\alpha\}_{\alpha \in A}$. For every $p \in M$, choose a neighborhood N_p of p that admits a homeomorphism $\varphi_p : N_p \to V_p$ where V_p is an open subset of $\mathbb{R}^n$. Then for every $U_\alpha \in \mathcal{T}$ that contains p, the set $\varphi_p(U_\alpha \cap N_p)$ is open in $\mathbb{R}^n$, so there exists an open ball $B \subseteq \varphi_p(U_\alpha \cap N_p)$ around $\varphi_p(p)$; let $B_{p,\alpha} = \varphi_p^{-1}(B) \subseteq U_\alpha \cap N_p$. Since $\varphi_p|_{B_{p,\alpha}} : B_{p,\alpha} \to B$ is a homeomorphism, $B_{p,\alpha}$ is a coordinate ball. By our construction, the set $\mathcal{B} = \{B_{p,\alpha} : (p, \alpha) \in M \times A \text{ and } p \in U_\alpha\}$ is a basis for M. □

Theorem 40. *[Problem 2-24] Suppose X is locally Euclidean of dimension n, and $f : X \to Y$ is a surjective local homeomorphism. Then Y is also locally Euclidean of dimension n.*

Proof. Let $y \in Y$ so that $y = f(x)$ for some $x \in X$. There exists a neighborhood N of x and homeomorphism $\varphi : N \to U$ where U is open in $\mathbb{R}^n$, and there exists a neighborhood N' of x such that $f(N')$ is open and $f|_{N'} : N' \to f(N')$ is a homeomorphism. Then $f(N \cap N')$ is a neighborhood of y and

$$\varphi|_{N \cap N'} \circ (f|_{N \cap N'})^{-1} : f(N \cap N') \to \varphi(N \cap N')$$

is a homeomorphism from a neighborhood of y to an open subset of $\mathbb{R}^n$. □

Theorem 41. *[Problem 2-25] Suppose M is an n-dimensional manifold with boundary. Then* $\operatorname{Int} M$ *is an n-manifold and ∂M is an $(n-1)$-manifold (without boundary).*

Proof. If $p \in \operatorname{Int} M$ then for some homeomorphism $\varphi : N \to U$ where $p \in N$ and U is an open subset of $\mathbb{H}^n$ we have $\varphi(p) \in \operatorname{Int} \mathbb{H}^n$. There exists an open ball $B \subseteq \operatorname{Int} \mathbb{H}^n \cap U$ around $\varphi(p)$, so $\varphi|_{\varphi^{-1}(B)}$ is a homeomorphism from the neighborhood $\varphi^{-1}(B)$ of p to the open subset B of $\mathbb{R}^n$. This shows that $\operatorname{Int} M$ is an n-manifold. Similarly, if $p \in \partial M$ then for some homeomorphism $\varphi : N \to U$ where $p \in N$ and U is an open subset of $\mathbb{H}^n$ we have $\varphi(p) \in \partial \mathbb{H}^n$. Let $\psi : \partial \mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection onto $\mathbb{R}^{n-1}$ that discards the last coordinate. There exists an open ball B in $\mathbb{R}^{n-1}$ around $\psi(\varphi(p))$ such that

$B \subseteq \psi(U \cap \partial\mathbb{H}^n)$, so some restriction of $\psi \circ \varphi$ is a homeomorphism from a neighborhood of p to B. This shows that ∂M is an $(n-1)$-manifold. □

Chapter 3. New Spaces from Old

Theorem 42. *[Exercise 3.1] Let X be a space and let $A \subseteq X$ be any subset. Define*

$$\mathcal{T}_A = \{U \subseteq A : U = A \cap V \text{ for some open set } V \subseteq X\}.$$

Then $\mathcal{T}_A$ is a topology on A.

Proof. It is clear that $\emptyset, A \in \mathcal{T}_A$. Let $\{U_\alpha\}_{\alpha \in A}$ be a subset of $\mathcal{T}_A$; for each α we have $U_\alpha = A \cap V_\alpha$ for set V_α open in X. Then

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} A \cap V_\alpha = A \cap \bigcup_{\alpha \in A} V_\alpha$$

which is an element of $\mathcal{T}_A$ since $\bigcup_{\alpha \in A} V_\alpha$ is open in X. Finally, if $U_1, \ldots, U_n \in \mathcal{T}_A$ then for each i we have $U_i = A \cap V_i$ for some V_i open in X, so

$$U_1 \cap \cdots \cap U_n = A \cap (V_1 \cap \cdots \cap V_n)$$

which is an element of $\mathcal{T}_A$ since $V_1 \cap \cdots \cap V_n$ is open in X. □

Theorem 43. *[Exercise 3.3] Let M be a metric space, and let $A \subseteq M$ be any subset. Then the subspace topology on A is the same as the metric topology obtained by restricting the metric of M to points in A.*

Proof. Denote the open ball in a metric space X of radius r around a point x by $B_r^{(X)}(x)$. To prove the result, it suffices to show that the two topologies have a common basis. Let U be a set in $\mathcal{T}_A$, so $U = A \cap V$ for some V open in M. If $p \in U$ then there exists some open ball $B_r^{(X)}(p)$ contained in V, and by definition $A \cap B_r^{(X)}(p) = B_r^{(A)}(p)$ is an open ball around p contained in A. This shows that the set of all open balls $\left\{B_r^{(A)}(p)\right\}$ is a basis for both topologies. □

Theorem 44. *[Exercise 3.12] Let A be a subspace of a topological space X.*

1. *The closed subsets of A are precisely the intersections of A with closed subsets of X.*
2. *If $B \subseteq A \subseteq X$, B is open in A, and A is open in X, then B is open in X.*
3. *If X is Hausdorff then A is Hausdorff.*
4. *If X is second countable then A is second countable.*

Proof. If E is closed in A then $A \setminus E = A \cap U$ for some U open in X, so $E = A \setminus (A \cap U) = A \cap (X \setminus U)$, where $X \setminus U$ is closed in X. The converse is similar. This proves (1). For part (2), we have $B = A \cap U$ for some U open in X, so B is open in X since A is open in X. Part (3) is obvious, and part (4) follows from part (b) of Proposition 3.11. □

Theorem 45. *[Exercise 3.25] Let $X_1, \dots, X_n$ be topological spaces. Define the set*

$$\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is open in } X_i,\ i = 1, \dots, n\}.$$

Then $\mathcal{B}$ is a basis for $X_1 \times \cdots \times X_n$.

Proof. It is clear that $\bigcup \mathcal{B} = X_1 \times \cdots \times X_n$. Let $U = U_1 \times \cdots \times U_n$ and $U' = U_1' \times \cdots \times U_n'$ be elements of $\mathcal{B}$. Since

$$U \cap U' = (U_1 \cap U_1') \times \cdots \times (U_n \cap U_n')$$

and each $U_i \cap U_i'$ is open in X_i, we have $U \cap U' \in \mathcal{B}$. This proves that $\mathcal{B}$ is a basis. □

Theorem 46. *[Exercise 3.26] The product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the same as the metric topology induced by the Euclidean distance function.*

Proof. It suffices to show that the set of all open balls forms a basis for the product topology on $\mathbb{R}^n$. Note first that every open ball is open in the product topology. If $U \subseteq \mathbb{R}^n$ is open in the product topology and $p = (p_1, \dots, p_n) \in U$, then there exist sets $U_1, \dots, U_n$ open in $\mathbb{R}$ such that $p \in U_1 \times \cdots \times U_n$ and $U_1 \times \cdots \times U_n \subseteq U$. For each i, choose an open ball $(p - r_i, p + r_i)$ contained in U_i. Let $r = \min\{r_1, \dots, r_n\}$; then $B_r(p)$ is an open ball around p that is contained in U. This proves that the set of all open balls is a basis for the product topology. □

Theorem 47. *[Exercise 3.32] Let $X_1, \dots, X_n$ be topological spaces.*

(1) *The projection maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are all continuous and open maps. In particular, a set $U_1 \times \cdots \times U_n$ is open in $X_1 \times \cdots \times X_n$ if and only if every U_i is open in X_i.*
(2) *The product topology is "associative" in the sense that the three product topologies $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$, and $X_1 \times (X_2 \times X_3)$ on the set $X_1 \times X_2 \times X_3$ are all equal.*
(3) *For any i and any points $x_j \in X_j$, $j \neq i$, the map $f_i : X_i \to X_1 \times \cdots \times X_n$ given by*

$$f_i(x) = (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

is a topological embedding of X_i into the product space.
(4) *If for each i, $\mathcal{B}_i$ is a basis for the topology of X_i, then the set*

$$\mathcal{B} = \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i\}$$

is a basis for the product topology on $X_1 \times \cdots \times X_n$.

(5) *If A_i is a subspace of X_i for $i = 1, \dots, n$, the product topology and the subspace topology on $A_1 \times \cdots \times A_n \subseteq X_1 \times \cdots \times X_n$ are equal.*
(6) *If each X_i is Hausdorff, so is $X_1 \times \cdots \times X_n$.*
(7) *If each X_i is second countable, so is $X_1 \times \cdots \times X_n$.*

Proof. The first part of (1) follows by taking $B = X_1 \times \cdots \times X_n$ and $f = \mathrm{Id}_B$ in Theorem 3.27. To show that π_i is an open map, let $U_1 \times \cdots \times U_n$ be an element of the standard basis of $X_1 \times \cdots \times X_n$. Since $\pi_i(U_1 \times \cdots \times U_n) = U_i$ which is open, it follows that $\pi_i(U)$ is open for any open set U. For (3), it suffices to check that $f_i^{-1}(U_1 \times \cdots \times U_n)$ is open whenever each U_i is open in X_i. If $x_j \in U_j$ for all $j \neq i$ then $f_i^{-1}(U_1 \times \cdots \times U_n) = U_i$, which is open. Otherwise, $f_i^{-1}(U_1 \times \cdots \times U_n) = \emptyset$, which is also open. For (4), it is clear that every set in $\mathcal{B}$ is open. Let U be an open set in $X_1 \times \cdots \times X_n$ and let $x = (x_1, \dots, x_n) \in U$. There exists some $U_1 \times \cdots \times U_n \subseteq U$ such that $x \in U_1 \times \cdots \times U_n$ and each U_i is open in X_i. Since $\mathcal{B}_i$ is a basis for X_i, there exists a $B_i \in \mathcal{B}_i$ with $x_i \in B_i$ and $B_i \subseteq U_i$. Therefore $x \in B_1 \times \cdots \times B_n$ and $B_1 \times \cdots \times B_n \subseteq U_1 \times \cdots \times U_n$, which proves that $\mathcal{B}$ is a basis for $X_1 \times \cdots \times X_n$.

For (5), it suffices to show that the basis

$$\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is open in } A_i,\ i = 1, \dots, n\}$$

for the product topology is also a basis for the subspace topology on $A_1 \times \cdots \times A_n$. First, note that if $U_1 \times \cdots \times U_n \in \mathcal{B}$ then for each i we have $U_i = A_i \cap V_i$ for some V_i open in X_i, so

$$U_1 \times \cdots \times U_n = (A_1 \times \cdots \times A_n) \cap (V_1 \times \cdots \times V_n)$$

which is open in the subspace topology. Now let $U \subseteq A_1 \times \cdots \times A_n$ be an open set in the subspace topology and let $x \in U$. We have $U = (A_1 \times \cdots \times A_n) \cap V$ for some V open in $X_1 \times \cdots \times X_n$. There exists some $V_1 \times \cdots \times V_n \subseteq V$ such that $x \in V_1 \times \cdots \times V_n$ and each V_i is open in X_i, so

$$(A_1 \cap V_1) \times \cdots \times (A_n \cap V_n) = (A_1 \times \cdots \times A_n) \cap (V_1 \times \cdots \times V_n) \subseteq U$$

is an element of $\mathcal{B}$ that contains x. This proves that $\mathcal{B}$ is a basis for the subspace topology.

For (6), let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be distinct points in $X_1 \times \cdots \times X_n$. We have $x_i \neq y_i$ for some i, so there exist neighborhoods U_1 of x_i and U_2 of y_i such that $U_1 \cap U_2 = \emptyset$. Then the sets $X_1 \times \cdots \times X_{i-1} \times U_j \times X_{i+1} \times \cdots \times X_n$ for $j = 1, 2$ are disjoint neighborhoods of x and y respectively. Part (7) follows from (4). $\square$

Theorem 48. *Let $X_1, \dots, X_n$ be topological spaces and let A_i be a subset of X_i for each i.*

(1) *If each A_i is closed in X_i then $A_1 \times \cdots \times A_n$ is closed in $X_1 \times \cdots \times X_n$.*
(2) $\overline{A_1 \times \cdots \times A_n} = \overline{A_1} \times \cdots \times \overline{A_n}$.

Proof. The projection maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are continuous, so $\pi_i^{-1}(A_i) = X_1 \times \cdots \times A_i \times \cdots \times X_n$ is closed in $X_1 \times \cdots \times X_n$ and

$$A_1 \times \cdots \times A_n = \bigcap_{i=1}^{n} \pi_i^{-1}(A_i)$$

is closed. For (2), it is clear that $\overline{A_1} \times \cdots \times \overline{A_n}$ is a closed set containing $A_1 \times \cdots \times A_n$, so $\overline{A_1 \times \cdots \times A_n} \subseteq \overline{A_1} \times \cdots \times \overline{A_n}$. Now let $x = (x_1, \ldots, x_n) \in \overline{A_1} \times \cdots \times \overline{A_n}$, let U be a neighborhood of x, and let $U_1 \times \cdots \times U_n \subseteq U$ be a neighborhood of x with each U_i open in X_i. Since U_i is a neighborhood of $x_i \in \overline{A_i}$, it contains a point $x_i' \neq x_i$ in A_i. Thus $x' = (x_1', \ldots, x_n')$ is a point of $A_1 \times \cdots \times A_n$ in U not equal to x, and x is a limit point of $A_1 \times \cdots \times A_n$. Since the closure of a set contains its limit points, $\overline{A_1} \times \cdots \times \overline{A_n} \subseteq \overline{A_1 \times \cdots \times A_n}$. □

Theorem 49. *[Exercise 3.43] Suppose we are given an indexed collection of nonempty topological spaces $\{X_\alpha\}_{\alpha \in A}$. Declare a subset of the disjoint union $X = \coprod_{\alpha \in A} X_\alpha$ to be open if and only if its intersection with each X_α is open.*

(1) *This is a topology on X, called the* ***disjoint union topology****.*
(2) *A subset of the disjoint union is closed if and only if its intersection with each X_α is closed.*
(3) *If each X_α is an n-manifold, then the disjoint union X is an n-manifold if and only if the index set A is countable.*

Proof. Let E be a closed subset of X. Since $X \setminus E$ is open, each $(X \setminus E) \cap X_\alpha = X_\alpha \setminus E$ is open, so the intersection of E with X_α is closed. The converse is similar. For (3), suppose that X is an n-manifold. Since X has a countable base $\mathcal{B}$ and each X_α is nonempty and open, there is a surjection from $\mathcal{B}$ to A, which shows that A is countable.

Conversely, suppose that A is countable. Let $x_1, x_2 \in X$ be distinct points. If $x_1, x_2 \in X_\alpha$ for some α then there exists neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$ since X_α is Hausdorff, and if $x_1 \in X_\alpha$, $x_2 \in X_\beta$ for $\alpha \neq \beta$ then X_α and X_β are neighborhoods of x_1 and x_2 respectively with $X_\alpha \cap X_\beta = \emptyset$. Therefore X is Hausdorff. Next, we prove that X is second countable. For each X_α, let $\mathcal{B}_\alpha$ be a countable base for X_α, and let $\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha$, which is countable. If U is an open set in X and $p \in U$ then $p \in X_\alpha \cap U$ for some α, so there exists some $B \in \mathcal{B}_\alpha$ with $p \in B$ and $B \subseteq X_\alpha \cap U$. Since $B \in \mathcal{B}$ and $B \subseteq U$, this shows that $\mathcal{B}$ is a countable base for X. Finally, it is easy to see that X is locally Euclidean of dimension n since each X_α is locally Euclidean of dimension n. Therefore X is an n-manifold. □

Theorem 50. *[Exercise 3.46] Let $(X, \mathcal{T})$ be a topological space, Y be any set, and $\pi : X \to Y$ be a surjective map. Then*

$$\mathcal{T}_Y = \left\{ U \subseteq Y : \pi^{-1}(U) \in \mathcal{T} \right\}$$

is a topology on Y.

Proof. Since $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(Y) = X$, we have $\emptyset, X \in \mathcal{T}_Y$. If $\{U_\alpha\}_{\alpha\in A}$ is a collection of sets in $\mathcal{T}_Y$ then

$$\pi^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) = \bigcup_{\alpha\in A} \pi^{-1}(U_\alpha) \in \mathcal{T}$$

since each $\pi^{-1}(U_\alpha)$ is in $\mathcal{T}$, so $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}_Y$. Similarly, if $U_1, \ldots, U_n \in \mathcal{T}_Y$ then

$$\pi^{-1}(U_1 \cap \cdots \cap U_n) = \pi^{-1}(U_1) \cap \cdots \cap \pi^{-1}(U_n) \in \mathcal{T}$$

since each $\pi^{-1}(U_i)$ is in $\mathcal{T}$, so $U_1 \cap \cdots \cap U_n \in \mathcal{T}_Y$. □

Theorem 51. *[Exercise 3.61] A continuous surjective map $\pi : X \to Y$ is a quotient map if and only if it takes saturated open sets to open sets, or saturated closed sets to closed sets.*

Proof. Suppose that π takes saturated open sets to open sets. If U is open in Y then $\pi^{-1}(U)$ is open in X since π is continuous. Also, if $U \subseteq Y$ and $\pi^{-1}(U)$ is open in X then $U = \pi(\pi^{-1}(U))$ is open since $\pi^{-1}(U)$ is saturated. Conversely, if π is a quotient map and $U = \pi^{-1}(V)$ is a saturated open set then $\pi(U) = \pi(\pi^{-1}(V)) = V$, which is open. Finally, π takes saturated open sets to open sets if and only if π takes saturated closed sets to closed sets, since $\pi^{-1}(U) = X \setminus \pi^{-1}(Y \setminus U)$ for any U open in Y. □

Theorem 52. *[Exercise 3.63]*

(1) *Any composition of quotient maps is a quotient map.*
(2) *An injective quotient map is a homeomorphism.*
(3) *If $q : X \to Y$ is a quotient map, a subset $K \subseteq Y$ is closed if and only if $q^{-1}(K)$ is closed in X.*
(4) *If $q : X \to Y$ is a quotient map and $U \subseteq X$ is a saturated open or closed subset, then the restriction $q|_U : U \to q(U)$ is a quotient map.*
(5) *If $\{q_\alpha : X_\alpha \to Y_\alpha\}_{\alpha\in A}$ is an indexed family of quotient maps, then the map $q : \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$ whose restriction to each X_α is equal to q_α is a quotient map.*

Proof. For (3), we have $Y \setminus K$ open if and only if $q^{-1}(Y \setminus K) = X \setminus q^{-1}(K)$ is open. For (4), let $U = q^{-1}(V)$ be a saturated open set and consider $q|_U : U \to q(U)$. If $A \subseteq q(U)$ then

$$(q|_U)^{-1}(A) = q^{-1}(A) \cap q^{-1}(V) = q^{-1}(A \cap q(U)) = q^{-1}(A),$$

so A is open if and only if $(q|_U)^{-1}(A) = q^{-1}(A)$ is open since q is a quotient map. The case for the restriction of q to a closed set is similar. For (5), let $U \subseteq \coprod_\alpha Y_\alpha$. We have

$$U \underset{\text{open}}{\subseteq} \coprod_\alpha Y_\alpha \Leftrightarrow U \cap Y_\alpha \underset{\text{open}}{\subseteq} Y_\alpha \text{ for every } \alpha \in A$$

$$\Leftrightarrow q_\alpha^{-1}(U) \underset{\text{open}}{\subseteq} X_\alpha \text{ for every } \alpha \in A$$
$$\Leftrightarrow q^{-1}(U) \cap X_\alpha \underset{\text{open}}{\subseteq} X_\alpha \text{ for every } \alpha \in A$$
$$\Leftrightarrow q^{-1}(U) \underset{\text{open}}{\subseteq} \coprod_\alpha X_\alpha.$$

□

Theorem 53. *[Exercise 3.85] Any subgroup of a topological group is a topological group with the subspace topology. Any finite product of topological groups is a topological group with the direct product group structure and the product topology.*

Proof. Let G be a topological group with operations $\mu : G \times G \to G$ and $\iota : G \to G$, and let H be a subgroup of G. Since $H \times H$ is a subspace of $G \times G$ and H is a subspace of G, the restrictions $\mu|_{H\times H}$ and $\iota|_H$ are continuous by Corollary 3.10. This shows that H is also a topological group with the subspace topology. Now let $G_1, \ldots, G_n$ be topological groups with operations $\mu_i : G_i \times G_i \to G_i$ and $\iota_i : G_i \to G_i$ for $i = 1, \ldots, n$. We want to show that the maps

$$\begin{aligned} \mu : (G_1 \times \cdots \times G_n) \times (G_1 \times \cdots \times G_n) &\to G_1 \times \cdots \times G_n \\ ((g_1, \ldots, g_n), (g_1', \ldots, g_n')) &\mapsto (g_1 g_1', \ldots, g_n g_n') \end{aligned}$$

and

$$\begin{aligned} \iota : G_1 \times \cdots \times G_n &\to G_1 \times \cdots \times G_n \\ (g_1, \ldots, g_n) &\mapsto (g_1^{-1}, \ldots, g_n^{-1}) \end{aligned}$$

are continuous. But

$$\begin{aligned} \varphi : (G_1 \times \cdots \times G_n) \times (G_1 \times \cdots \times G_n) &\to (G_1 \times G_1) \times \cdots \times (G_n \times G_n) \\ ((g_1, \ldots, g_n), (g_1', \ldots, g_n')) &\mapsto ((g_1, g_1'), \ldots, (g_n, g_n')) \end{aligned}$$

is continuous, being a simple rearrangement, and Proposition 3.33 shows that

$$\mu_1 \times \cdots \times \mu_n : (G_1 \times G_1) \times \cdots \times (G_n \times G_n) \to G_1 \times \cdots \times G_n$$

is continuous. Therefore μ is continuous since $\mu = (\mu_1 \times \cdots \times \mu_n) \circ \varphi$. Finally, $\iota = \iota_1 \times \cdots \times \iota_n$ is continuous by Proposition 3.33. □

Example 54. [Problem 3-3] By considering the space $X = [0, 1] \subseteq \mathbb{R}$ and the sets $A_0 = \{0\}$, $A_i = [1/(i + 1), 1/i]$ for $i = 1, 2, \ldots$, show that the gluing lemma (Lemma 3.23) is false if $\{A_1, \ldots, A_k\}$ is replaced by an infinite sequence of closed sets.

Define the sequence of maps $f_i : A_i \to \mathbb{R}$ by setting $f_0(0) = 1$ and $f_i(x) = 0$ for all $i \geq 1$ and $x \in A_i$. By the gluing lemma, there is a unique continuous map $f : X \to \mathbb{R}$ such that $f(0) = 1$ and $f(x) = 0$ for all $x \in (0, 1]$. However, this map is clearly not continuous at 0.

Theorem 55. *[Problem 3-4] Any closed ball in $\mathbb{R}^n$ is an n-dimensional manifold with boundary.*

Proof. Let $\overline{\mathbb{B}^n} = \overline{B_1(0)}$ be the closed unit ball in $\mathbb{R}^n$, let $N = (0, \dots, 0, 1)$ be the "north pole", and let $T = \{0\}^{n-1} \times [0, 1]$ be the vertical line connecting 0 and N. Define $\sigma : \overline{\mathbb{B}^n} \setminus T \to \mathbb{H}^n$ given by

$$\sigma(x_1, \dots, x_n) = \left(\frac{x_1}{\|x\| - x_n}, \dots, \frac{x_{n-1}}{\|x\| - x_n}, \|x\|^{-1} - 1 \right),$$

with its inverse given by

$$\sigma^{-1}(u_1, \dots, u_n) = \frac{1}{(u_n + 1)(\|\widetilde{u}\|^2 + 1)}(2u_1, \dots, 2u_{n-1}, \|\widetilde{u}\|^2 - 1)$$

where $\widetilde{u} = (u_1, \dots, u_{n-1})$. Since both maps are continuous, this proves that $\overline{\mathbb{B}^n} \setminus T$ is homeomorphic to $\mathbb{H}^n$. Furthermore, we can repeat the same argument by taking the vertical line $T = \{0\}^{n-1} \times [-1, 0]$ instead, giving us Euclidean neighborhoods of every point in $\overline{\mathbb{B}^n}$ except for 0. But the identity map on the open unit ball $\mathbb{B}^n$ is a suitable coordinate chart around 0, so this proves that $\overline{\mathbb{B}^n}$ is an n-manifold with boundary. □

Theorem 56. *[Problem 3-5] A finite product of open maps is open. A finite product of closed maps need not be closed.*

Proof. Let $f_i : X_i \to Y_i$, $i = 1, \dots, n$, be a set of open maps and write $f = f_1 \times \cdots \times f_n$. Let $U \subseteq X_1 \times \cdots \times X_n$ be an open set; we want to show that $f(U)$ is open. If $y \in f(U)$ then $y = f(x)$ for some $x = (x_1, \dots, x_n) \in U$. There exists some $U_1 \times \cdots \times U_n \subseteq U$ such that $x \in U_1 \times \cdots \times U_n$ and each U_i is open in X_i. For every i, $f_i(U_i)$ is open since f_i is an open map, so $y \in f_1(U_1) \times \cdots \times f_1(U_n) \subseteq f(U)$. This proves that $f(U)$ is open. However, a finite product of closed maps need not be closed. Let $D = \{0\}$ be a discrete space, let $f : \mathbb{R} \to \mathbb{R}$ be the identity on $\mathbb{R}$ and let $g : \mathbb{R} \to D$ be given by $g(x) = 0$. Then f and g are both closed, but

$$(f \times g)(\{(x, y) : xy = 1\}) = (\mathbb{R} \setminus \{0\}) \times \{0\},$$

which is not closed. □

Theorem 57. *[Problem 3-6] Let X be a topological space. The* ***diagonal*** *of $X \times X$ is the subset $\triangle = \{(x, x) : x \in X\} \subseteq X \times X$. Then X is Hausdorff if and only if $\triangle$ is closed in $X \times X$.*

Proof. Suppose X is Hausdorff and let $(x_1, x_2) \in (X \times X) \setminus \triangle$. Since $x_1 \neq x_2$, there exist neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. This implies that $(U_1 \times U_2) \cap \triangle = \emptyset$, so $U_1 \times U_2$ is a neighborhood of (x_1, x_2) with $U_1 \times U_2 \subseteq (X \times X) \setminus \triangle$. This shows that $(X \times X) \setminus \triangle$ is open. Conversely, suppose that $(X \times X) \setminus \triangle$ is open and let x_1, x_2 be distinct points in X. Then $(x_1, x_2) \in (X \times X) \setminus \triangle$, so there exist open sets $U_1, U_2 \subseteq X$ such that $(x_1, x_2) \in U_1 \times U_2 \subseteq (X \times X) \setminus \triangle$. Therefore $U_1 \cap U_2 = \emptyset$, which proves that X is Hausdorff. ☐

Theorem 58. *[Problem 3-7] Let $M = \mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ is the set $\mathbb{R}$ with the discrete topology.*

(1) *M is homeomorphic to the space X of Example 38.*
(2) *M is locally Euclidean and Hausdorff, but not second countable.*

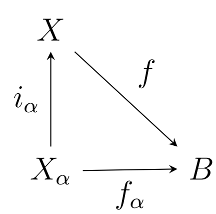
Proof. Let $U \times V \subseteq M$ where V is open in $\mathbb{R}$. For each $u \in U$ we can write $\{u\} \times V = \bigcup_{\alpha \in A}(a_\alpha, b_\alpha)$, which is open in X. Therefore

$$U \times V = \bigcup_{u \in U} \{u\} \times V$$

is open in X. Also, every element $\{(c, y) : a < y < b\}$ in the basis of X is clearly open in M. Part (1) then follows from Lemma 22. From Theorem 47 we know that M is locally Euclidean (of dimension 2) and Hausdorff. To show that M is not second countable, suppose $\mathcal{B}$ is a countable basis for M. Consider the collection $\mathcal{C}$ of sets of the form $\{x\} \times \mathbb{R}$ for $x \in \mathbb{R}$, which are all disjoint and open in M. By Theorem 34, $\mathcal{C}$ is countable. But $\{x\} \times \mathbb{R} \mapsto x$ is a surjection from $\mathcal{C}$ to $\mathbb{R}$, which contradicts the uncountability of $\mathbb{R}$. ☐

Theorem 59. *[Problem 3-10]*

(1) *(Characteristic Property of Disjoint Union Topologies) Let $X = \coprod_{\alpha \in A} X_\alpha$ be a disjoint union space. For any topological space B, a map $f : X \to B$ is continuous if and only if each $f_\alpha = f \circ i_\alpha$ is continuous, where $i_\alpha : X_\alpha \to X$ is the canonical injection:*



(2) *(Uniqueness of the Disjoint Union Topology) Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. The disjoint union topology on $\coprod_{\alpha \in A} X_\alpha$ is the unique topology that satisfies the characteristic property.*

Proof. We have

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow f^{-1}(U) \underset{\text{open}}{\subseteq} X \text{ for all } U \underset{\text{open}}{\subseteq} B \\ &\Leftrightarrow f^{-1}(U) \cap X_\alpha \underset{\text{open}}{\subseteq} X_\alpha \text{ for all } \alpha \in A \text{ and } U \underset{\text{open}}{\subseteq} B \end{aligned}$$

$$\Leftrightarrow f_\alpha^{-1}(U) \underset{\text{open}}{\subseteq} X_\alpha \text{ for all } \alpha \in A \text{ and } U \underset{\text{open}}{\subseteq} B$$
$$\Leftrightarrow f_\alpha \text{ is continuous for all } \alpha \in A,$$

which proves (1). For (2), suppose that $X_g = \coprod_{\alpha \in A} X_\alpha$ has some other topology that satisfies the characteristic property, and write X_d for the usual disjoint union space on $\coprod_{\alpha \in A} X_\alpha$. By applying the characteristic property to the identity map, we see that every injection i_α into either X_g or X_d is continuous. Setting $B = X_g$ and then $B = X_d$ shows that the identity map from the disjoint union topology to the given topology is a homeomorphism. Therefore the topologies on X_g and X_d are equal. □

Theorem 60. *Let $X_1, \ldots, X_n, Y$ be topological spaces. Then the identity map*

$$\iota : (X_1 \amalg \cdots \amalg X_n) \times Y \to (X_1 \times Y) \amalg \cdots \amalg (X_n \times Y)$$

is a homeomorphism.

Proof. Let U be open in $(X_1 \times Y) \amalg \cdots \amalg (X_n \times Y)$; then $U \cap (X_i \times Y)$ is open in $X_i \times Y$ for every i. Let $(x, y) \in U$ so that $x \in X_i$ for some i. Since $x \in U \cap (X_i \times Y)$, we have $x \in V \times W \subseteq U \cap (X_i \times Y)$ for some V open in X_i and some W open in Y. But $V \times W$ is also open in $(X_1 \amalg \cdots \amalg X_n) \times Y$, which shows that U is open in $(X_1 \amalg \cdots \amalg X_n) \times Y$. Conversely, let U be open in $(X_1 \amalg \cdots \amalg X_n) \times Y$ and let $(x, y) \in U$. We have $(x, y) \in V \times W \subseteq U$ for some V open in $X_1 \amalg \cdots \amalg X_n$ and some W open in Y. Since $(V \times W) \cap (X_i \times Y) = (V \cap X_i) \times W$ is open in $X_i \times Y$ for every i, $V \times W \subseteq U$ is a neighborhood of (x, y) in $(X_1 \times Y) \amalg \cdots \amalg (X_n \times Y)$. This shows that U is open in $(X_1 \times Y) \amalg \cdots \amalg (X_n \times Y)$. □

Theorem 61. *If $X_1, \ldots, X_k$ are nonempty topological spaces then the projections $\pi_i : X_1 \times \cdots \times X_k \to X_i$ are quotient maps.*

Proof. This follows from the fact that the projections are surjective, continuous and open maps. See Theorem 47. □

Example 62. [Problem 3-11] Proposition 3.62(d) showed that the restriction of a quotient map to a saturated open set is still a quotient map. Show that the "saturated" hypothesis is necessary, by giving an example of a quotient map $f : X \to Y$ and an open subset $U \subseteq X$ such that $f|_U$ is surjective but not a quotient map.

The map $f : [0, 1] \to \mathbb{S}^1$ given by $f(s) = e^{2\pi i s}$ is a quotient map, but $f|_{[0,1)}$ is not a quotient map as Example 3.66 shows.

Theorem 63. *[Problem 3-14] Real projective space $\mathbb{P}^n$ is an n-manifold.*

Proof. Denote a line $\{\lambda(p_1, \ldots, p_{n+1}) : \lambda \in \mathbb{R}\} \in \mathbb{P}^n$ by $[p_1, \ldots, p_{n+1}]$. Given some k, let $V_k = \{[p_1, \ldots, p_{n+1}] \in \mathbb{P}^n : p_k \neq 0\}$ be the lines of $\mathbb{P}^n$ that are not parallel to the plane

$x_k = 1$. Define $\pi : \mathbb{P}^n \setminus V_k \to \mathbb{R}^n$ by

$$\pi([p_1, \ldots, p_{n+1}]) = p_k^{-1}(p_1, \ldots, \widehat{p_k}, \ldots, p_{n+1})$$

where p_k is omitted. This map is well-defined since $[p_1, \ldots, p_{n+1}] = [q_1, \ldots, q_{n+1}]$ implies that $p_i = \lambda q_i$ for every i. Furthermore, π is a homeomorphism. By choosing at least two different values of k, we can find a coordinate ball around every element of $\mathbb{P}^n$. This shows that $\mathbb{P}^n$ is an n-manifold. □

Theorem 64. *[Problem 3-15] Let $\mathbb{CP}^n$ denote the set of all 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$, called **n-dimensional complex projective space**. Topologize $\mathbb{CP}^n$ as the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ is the group of nonzero complex numbers acting by scalar multiplication. Then $\mathbb{CP}^n$ is a $2n$-manifold.*

Proof. Proceed as in Theorem 63 to show that each $\mathbb{P}^n \setminus V_k$ is homeomorphic to $\mathbb{C}^n$. Since $\mathbb{C}^n$ is a $2n$-manifold, this proves that $\mathbb{CP}^n$ is a $2n$-manifold. □

Theorem 65. *[Problem 3-16] Let X be the subset $\mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$ of $\mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. Then the quotient space $X/\sim$ is locally Euclidean and second countable, but not Hausdorff. (This space is called the **line with two origins**.)*

Proof. Let $p \in X/\sim$. If $p \neq [(0,0)]$ and $p \neq [(0,1)]$ then there is clearly a neighborhood N of p homeomorphic to some open interval in $\mathbb{R}$. If $p = [(0,0)]$ then let $N = \{[(x,0)] : -1 < x < 1\}$ and define $\varphi : N \to (-1,1)$ by $[(x,0)] \mapsto x$. It is easy to check that this map is a well-defined homeomorphism. So $X/\sim$ is locally Euclidean and therefore second countable by Proposition 3.56. But $X/\sim$ is not Hausdorff, since any two neighborhoods of $[(0,0)]$ and $[(0,1)]$ respectively must contain a common point. □

Theorem 66. *[Problem 3-19] If G is a topological group and $H \subseteq G$ is a subgroup, then $\overline{H}$ is also a subgroup.*

Proof. Let $\mu : G \times G \to G$ and $\iota : G \to G$ be the product and inverse maps respectively. Since $\iota(H) \subseteq H$ we have $\iota(\overline{H}) \subseteq \overline{H}$ by Lemma 21. Similarly, $\mu(H \times H) \subseteq H$ implies that $\mu(\overline{H \times H}) \subseteq \overline{H}$, and $\overline{H \times H} = \overline{H} \times \overline{H}$ by Theorem 48. □

Theorem 67. *[Problem 3-20] If G is a group that is also a topological space, then G is a topological group if and only if the map $\theta : G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ is continuous.*

Proof. Since $\theta(x,y) = \mu(x, \iota(y))$, θ is continuous if G is a topological group. Conversely, since $\iota(x) = \theta(1, x)$ and $\mu(x, y) = \theta(x, \iota(y))$, G is a topological group if θ is continuous. □

Theorem 68. *[Problem 3-21] Let G be a topological group and $\Gamma \subseteq G$ be a subgroup.*

(1) *For any $g \in G$, the left translation $L_g : G \to G$ passes to the quotient G/Γ and defines a homeomorphism of G/Γ with itself.*
(2) *A topological space X is said to be* ***homogeneous*** *if for any $x, y \in X$, there is a homeomorphism $\varphi : X \to X$ taking x to y. Every coset space is homogeneous.*

Proof. Let $\pi : G \to G/\Gamma$ given by $g \mapsto g\Gamma$ be the quotient map. Since $\pi \circ L_g : G \to G/\Gamma$ satisfies $(\pi \circ L_g)(g') = gg'\Gamma$ and is constant on the fibers of π, there exists a unique continuous map $\widetilde{L}_g : G/\Gamma \to G/\Gamma$ satisfying $\widetilde{L}_g \circ \pi = \pi \circ L_g$. Furthermore, $\widetilde{L}_{g^{-1}}$ is a continuous inverse of $\widetilde{L}_g$ since

$$\widetilde{L}_{g^{-1}} \circ \widetilde{L}_g \circ \pi = \widetilde{L}_{g^{-1}} \circ \pi \circ L_g = \pi \circ L_{g^{-1}} \circ L_g = \pi$$

and similarly $\widetilde{L}_g \circ \widetilde{L}_{g^{-1}} \circ \pi = \pi$. This shows that $\widetilde{L}_g$ is a homeomorphism of G/Γ with itself. For (2), if $g\Gamma$ and $g'\Gamma$ are cosets in G/Γ then $\widetilde{L}_{g'g^{-1}}$ is a homeomorphism that takes $g\Gamma$ to $g'\Gamma$. □

Theorem 69. *[Problem 3-22] Let G be a topological group acting continuously on a topological space X.*

(1) *The quotient map $\pi : X \to X/G$ is open.*
(2) *X/G is Hausdorff if and only if the orbit relation*

$$D = \{(x_1, x_2) \in X \times X : x_2 = g \cdot x_1 \text{ for some } g \in G\}$$

is closed in $X \times X$.

Proof. Let $U \subseteq X$ be open; we want to show that $\pi(U)$ is open, or equivalently, that $\pi^{-1}(\pi(U)) = \{g \cdot x : g \in G, x \in U\}$ is open. Since the group action $\alpha : G \times X \to X$ is continuous, $\alpha^{-1}(U)$ is open in $G \times X$. Let $\pi_2 : G \times X \to X$ be the canonical projection. If $g \cdot x \in \pi^{-1}(\pi(U))$ then $\alpha(g^{-1}, g \cdot x) = x \in U$, so $g \cdot x \in \pi_2(\alpha^{-1}(U))$. Conversely, if $x \in \pi_2(\alpha^{-1}(U))$ then $g \cdot x \in U$ for some $g \in G$, so $g^{-1} \cdot (g \cdot x) = x \in \pi^{-1}(\pi(U))$. This shows that $\pi^{-1}(\pi(U)) = \pi_2(\alpha^{-1}(U))$. But this set is open by Theorem 47. For (2), notice that D being closed is equivalent to $\triangle = \{(p, p) : p \in X/G\}$ being closed in X/G, and apply Theorem 57. □

Theorem 70. *[Problem 3-23] If Γ is a normal subgroup of the topological group G then the coset space G/Γ is a topological group.*

Proof. Let $\pi : G \to G/\Gamma$ be the quotient map (in the group sense). By Theorem 69, π is also a quotient map in the topology sense. Let $\mu : G \times G \to G$ and $\iota : G \to G$ be the product and inverse maps respectively. By Theorem 56, $\pi \times \pi : G \times G \to G/\Gamma \times G/\Gamma$ is a continuous, surjective and open map, and therefore a quotient map. Since Γ is normal in G, $\pi \circ \mu$ is constant on the fibers of $\pi \times \pi$ and there exists a unique continuous map $\widetilde{\mu} : G/\Gamma \times G/\Gamma \to G/\Gamma$ satisfying $\widetilde{\mu} \circ (\pi \times \pi) = \pi \circ \mu$. Similarly, there exists a unique continuous map $\widetilde{\iota} : G/\Gamma \to G/\Gamma$ satisfying $\widetilde{\iota} \circ \pi = \pi \circ \iota$. The group axioms are then easily checked for $\widetilde{\mu}$ and $\widetilde{\iota}$. ☐

Chapter 4. Connectedness and Compactness

Theorem 71. *In a topological space X, the path connectivity relation $\sim_p$ is an equivalence relation.*

Proof. For any point $p \in X$, the path $f : [0,1] \to X$ with $f(t) = p$ is a path from p to itself. If $f : [0,1] \to X$ is a path from p to q, then $g : [0,1] \to X$ given by $g(t) = f(1-t)$ is a path from q to p. Finally, if $f : [0,1] \to X$ is a path from p to q and $g : [0,1] \to X$ is a path from q to r, then by the gluing lemma the map $h : [0,1] \to X$ given by

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

is a path from p to r. ☐

Theorem 72. *[Exercise 4.22] Let X be any space.*

1. *Each path component is contained in a single component, and each component is a disjoint union of path components.*
2. *If $A \subseteq X$ is path-connected, then A is a contained in a single path component.*

Proof. (1) follows from the fact that every path component is also connected and Proposition 4.21. For (2), suppose B and C are path components both containing points of A. It follows from Theorem 71 that $B = C$. ☐

Theorem 73. *[Exercise 4.24] Every manifold is locally path-connected.*

Proof. This follows from Theorem 39. ☐

Theorem 74. *[Exercise 4.38] Let X be a compact space, and suppose $\{F_n\}$ is a countable collection of nonempty closed subsets of X that are **nested**, which means that $F_n \supseteq F_{n+1}$ for each n. Then $\bigcap_n F_n$ is nonempty.*

Proof. Suppose that $\bigcap_n F_n$ is empty. Then $X = \bigcup_n X \setminus F_n$, so there is a finite subcover $\{X \setminus F_{n_1}, \ldots, X \setminus F_{n_k}\}$ where we take $n_1 < \cdots < n_k$. But $\bigcup_{i=1}^k X \setminus F_{n_i} = X \setminus F_{n_k}$ since $F_n \supseteq F_{n+1}$ for each n, and this is a contradiction since F_{n_k} is nonempty. ☐

Theorem 75. *[Exercise 4.49] Every compact metric space is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a metric space M. There is some subsequence that converges to a point x, and it is well-known that a Cauchy sequence converges if it has a convergent subsequence. Therefore $x_n \to x$. ☐

Theorem 76. *[Exercise 4.67] Any finite product of locally compact spaces is locally compact.*

Proof. It suffices to show that if X, Y are locally compact then $X \times Y$ is locally compact. Let $(x, y) \in X \times Y$. By Proposition 4.63, there exist precompact neighborhoods U of x and V of y. Then $U \times V$ is a precompact neighborhood of (x, y), since $\overline{U \times V} = \overline{U} \times \overline{V}$ is compact. □

Theorem 77. *[Exercise 4.73] Suppose $\mathcal{A}$ is an* open *cover of X such that each element of $\mathcal{A}$ intersects only finitely many others. Then $\mathcal{A}$ is locally finite. This need not be true when the elements of $\mathcal{A}$ are not open.*

Proof. Let $x \in X$ and choose some $A \in \mathcal{A}$ such that $x \in A$. There are finitely many elements of $\mathcal{A}$ that intersect A, so A is the required neighborhood of x. For a counterexample when the elements of $\mathcal{A}$ are not required to be open, take $X = \mathbb{R}$ and $\mathcal{A} = \{\{x\} : x \in \mathbb{R}\}$. □

Theorem 78. *[Exercise 4.78,4.79]*

(1) *Every compact Hausdorff space is normal.*
(2) *Every closed subspace of a normal space is normal.*

Proof. Let X be a compact Hausdorff space and let A, B be closed subsets of X. Then A, B are compact, so by Lemma 4.34 there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. For (2), let X be a normal space and let E be closed in X. If A, B are closed subsets of E then there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. So $U \cap E$ and $V \cap E$ are disjoint open subsets of E such that $A \subseteq U \cap E$ and $B \subseteq V \cap E$. □

Theorem 79. *[Exercise 4.87] Every compact manifold with boundary is homeomorphic to a subset of some Euclidean space.*

Proof. If M is a compact manifold with boundary then the double $D(M)$ of M is a compact manifold (without boundary), and is homeomorphic to a subset of some Euclidean space. The restriction of this homeomorphism to M is the desired homeomorphism. □

Theorem 80. *[Problem 4-1]*

(1) *If U is any open subset of $\mathbb{R}$ and $x \in U$, then $U \setminus \{x\}$ is disconnected.*
(2) *For $n > 1$, $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$.*

Proof. Let $A = \{t \in \mathbb{R} : t < x\}$ and $B = \{t \in \mathbb{R} : t > x\}$; then $\{A \cap U, B \cap U\}$ is a separation of $U \setminus \{x\}$. This proves (1). Part (2) follows immediately. □

Theorem 81. *[Problem 4-2] A nonempty topological space cannot be both a* 1*-manifold and an* n*-manifold for any* $n > 1$*.*

Proof. Let M be a nonempty topological space that is both a 1-manifold and an n-manifold for some $n > 1$. Choose some $p \in M$ and let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homeomorphisms where U_1 and U_2 are neighborhoods of p, V_1 is open in $\mathbb{R}$, and V_2 is open in $\mathbb{R}^n$. Let B be an open ball around $\varphi_2(p)$ contained in $\varphi_2(U_1 \cap U_2)$. Then $W_1 = B \setminus \{\varphi_2(p)\}$ is homeomorphic to $W_2 = (\varphi_1 \circ \varphi_2^{-1})(B) \setminus \{\varphi_1(p)\}$, but W_2 is disconnected by Theorem 80 while W_1 is (path) connected. This is a contradiction. □

Theorem 82. *[Problem 4-3] Suppose* M *is a* 1*-dimensional manifold with boundary. Then the interior and boundary of* M *are disjoint.*

Proof. Suppose $p \in M$ is both an interior and boundary point. Choose coordinate charts (U, φ) and (V, ψ) such that U, V are neighborhoods of p, $\varphi(U)$ is open in $\operatorname{Int} \mathbb{H}^1$, $\psi(V)$ is open in $\mathbb{H}^1$, $\varphi(p) > 0$ and $\psi(p) = 0$. Let $W = U \cap V$; then $\varphi(W)$ is homeomorphic to $\psi(W)$. But this is impossible, for $\varphi(W) \setminus \{\varphi(p)\}$ is disconnected while $\psi(W) \setminus \{\psi(p)\}$ is connected. □

Theorem 83. *[Problem 4-4] The following topological spaces are not manifolds:*

(1) *The union of the* x*-axis and the* y*-axis in* $\mathbb{R}^2$*.*
(2) *The conical surface* $C \subseteq \mathbb{R}^3$ *defined by*

$$C = \{(x, y, z) : z^2 = x^2 + y^2\}$$

Proof. Let M be the union of the x-axis and the y-axis, and suppose that M is a manifold. By Theorem 20, M cannot be a 0-manifold. Now suppose M is a 1-manifold and let B be a coordinate ball around $(0, 0)$ with a homeomorphism $\varphi : B \to (a, b)$. Removing the point $\varphi(0, 0)$ from (a, b) produces two connected components, but removing $(0, 0)$ from B produces four connected components (the left, top, right and bottom parts of the cross shape). Therefore M must be a n-manifold for $n > 1$. But this implies that the positive x-axis is both a 1-manifold and an n-manifold, contradicting Theorem 81. Part (2) is similar, for there is no 2 dimensional coordinate ball around $(0, 0)$. □

Theorem 84. *[Problem 4-7] Suppose* $f : X \to Y$ *is a surjective local homeomorphism. If* X *is locally connected, locally path-connected, or locally compact, then* Y *has the same property.*

Proof. The result for the first two properties follow from Theorem 30. Suppose that X is locally compact and let $\mathcal{B}$ be a basis of precompact open sets in X. Let $y \in Y$ so that $y = f(x)$ for some $x \in X$. Let U be a neighborhood of x such that $f(U)$ is open and $f|_U : U \to f(U)$ is a homeomorphism. There exists some neighborhood $E \subseteq U$ of

x that is precompact, so $f(E)$ is a precompact neighborhood of y. This proves that Y is locally compact. □

Theorem 85. *[Problem 4-9] Any n-manifold is a disjoint union of countably many connected n-manifolds.*

Proof. Any n-manifold M is the disjoint union of its connected components, which are open by Proposition 4.25 and therefore are n-manifolds by Proposition 2.53. Furthermore, the collection of connected components of M is countable by Theorem 34. □

Example 86. [Problem 4-13] Let X be the topologist's sine curve (Example 4.17).

(1) Show that X is connected but not path-connected or locally connected.
(2) Determine the components and path components of X.

The topologist's sine curve is the union of the two sets

$$A = \{(x, y) : x = 0 \text{ and } y \in [-1, 1]\};$$
$$B = \{(x, y) : y = \sin(1/x) \text{ and } x \in (0, 1]\}.$$

As subsets of $\mathbb{R}^2$, since $\overline{B} = A \cup B = X$ and B is connected, Proposition 4.9 shows that X is connected. Suppose that X is path-connected and let $\gamma : [0, 1] \to X$ be path connecting the points $(0, 0)$ and $(2/\pi, 1)$. Choose a $\delta > 0$ such that $\|\gamma(t)\| < 1/2$ whenever $0 \leq t < \delta$. This is impossible, for $(0, 1)$ is a limit point of B. Also, X is not locally connected since any neighborhood of $(0, 0)$ is disconnected. The two path components of X are exactly A and B.

Theorem 87. *[Problem 4-15] Suppose G is a topological group.*

(1) *Every open subgroup of G is also closed.*
(2) *For any neighborhood U of 1, the subgroup $\langle U \rangle$ generated by U is open and closed in G.*
(3) *For any connected subset $U \subseteq G$ containing 1, $\langle U \rangle$ is connected.*
(4) *If G is connected, then every neighborhood of 1 generates G.*

Proof. Let $\mu : G \times G \to G$ and $\iota : G \to G$ be the group operations. For (1), let H be an open subgroup of G. Then every coset gH is open in G, and $G \backslash H = \bigcup_{g \in G \backslash H} gH$ is open. For (2), let $x \in \langle U \rangle$. Since $\mu(x, \cdot)$ is a homeomorphism, $\mu(x, U) \subseteq \langle U \rangle$ is a neighborhood of x. Then $\langle U \rangle = \bigcup_{x \in \langle U \rangle} \mu(x, U)$ is open. For (3), let $x \in \langle U \rangle$. Then $\mu(x, U) \subseteq \langle U \rangle$ is a connected set containing x, and $\mu(\mu(x, U), U) \subseteq \langle U \rangle$ is a connected set containing both x and 1. By Proposition 4.9, $\langle U \rangle = \bigcup_{x \in \langle U \rangle} \mu(\mu(x, U), U)$ is connected. For (4), if U is a neighborhood of 1 then $\langle U \rangle$ is both open and closed in G by (2). Therefore $\langle U \rangle = G$. □

Theorem 88. *Every σ-compact space is Lindelöf.*

Proof. Let X be a σ-compact space and let $\mathcal{A}$ be a countable collection of compact subsets that cover X. Let $\mathcal{U}$ be an open cover of X. Each $A \in \mathcal{A}$ is covered by finitely many sets from $\mathcal{U}$, so there is a countable subcover. $\square$

Theorem 89. *[Problem 4-16] A locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

Proof. Let M be a locally Euclidean Hausdorff space. If M is a topological manifold then Proposition 4.60 shows that there is a countable collection $\mathcal{B}$ of regular coordinate balls that cover M, i.e. $M = \bigcup_{B \in \mathcal{B}} \overline{B}$. Conversely, suppose that M is σ-compact. Then M is Lindelöf by Theorem 88, so the argument of Proposition 4.60 shows that M is second countable. $\square$

Theorem 90. *[Problem 4-17] Suppose M is a manifold of dimension $n \geq 1$, and $B \subseteq M$ is a regular coordinate ball. Then $M \setminus B$ is an n-manifold with boundary, whose boundary is homeomorphic to $\mathbb{S}^{n-1}$.*

Proof. It suffices to show that there is a coordinate ball around every point of ∂B. There exists a neighborhood B' of $\overline{B}$ and a homeomorphism $\varphi : B' \to B_{r'}(0)$ that takes B to $B_r(0)$ for some $r' > r > 0$. In particular, $B' \setminus B$ is homeomorphic to $B_{r'}(0) \setminus B_r(0)$. For any $x \in \partial B$ there is a homeomorphism $\psi : U \to V$ where U is a neighborhood of $\varphi(x)$ in $B_{r'}(0) \setminus B_r(0)$, V is open in $\mathbb{H}^n$, and $\psi(\varphi(x)) = 0$. Then $\psi \circ \varphi$ is the required coordinate map from a neighborhood of $x \in \partial B$ in $B' \setminus B$. The boundary of $M \setminus B$ is $\partial B \approx \partial B_r(0) \approx \mathbb{S}^{n-1}$. $\square$

Theorem 91. *[Problem 4-18] Let M_1 and M_2 be n-manifolds. For $i = 1, 2$, let $B_i \subseteq M_i$ be regular coordinate balls, and let $M_i' = M_i \setminus B_i$. Choose a homeomorphism $f : \partial M_2' \to \partial M_1'$ (such a homeomorphism exists by Problem 4-17). Let $M_1 \# M_2$ (called a* ***connected sum of*** *M_1* ***and*** *M_2) be the adjunction space $M_1' \cup_f M_2'$.*

1. *$M_1 \# M_2$ is an n-manifold (without boundary).*
2. *If M_1 and M_2 are connected and $n > 1$, then $M_1 \# M_2$ is connected.*
3. *If M_1 and M_2 are compact, then $M_1 \# M_2$ is compact.*

Proof. Part (1) follows from Theorem 3.79. If M_1 and M_2 are connected and $n > 1$ then M_1' and M_2' are still connected. Let $e : M_1' \to M_1' \cup_f M_2'$ and $f : M_2' \to M_1' \cup_f M_2'$ be the canonical embeddings. By Theorem 3.79 we have $e(M_1') \cap f(M_2') \neq \emptyset$, so $M_1 \# M_2$ is connected. If M_1 and M_2 are compact then M_1' and M_2' are closed and therefore compact. This implies that $M_1 \# M_2 = e(M_1') \cup f(M_2')$ is compact. $\square$

Theorem 92. *[Problem 4-19] Let $M_1 \# M_2$ be a connected sum of n-manifolds M_1 and M_2. There are open subsets $U_1, U_2 \subseteq M_1 \# M_2$ and points $p_i \in M_i$ such that $U_i \approx M_i \setminus \{p_i\}$, $U_1 \cap U_2 \approx \mathbb{R}^n \setminus \{0\}$, and $U_1 \cup U_2 = M_1 \# M_2$.*

Proof. For $i = 1, 2$, let $B_i \subseteq M_i$ be the regular coordinate ball around $p_i \in M_i$ and let $C_i \supseteq \overline{B_i}$ be the larger coordinate balls around p_i. Let $j_i : M_i \setminus B_i \to M_1 \# M_2$ be the injections. Take $U_1 = j_1(M_1 \setminus B_1) \cup j_2(C_2 \setminus B_2)$ and $U_2 = j_1(C_1 \setminus B_1) \cup j_2(M_2 \setminus B_2)$, noting that

$$\begin{aligned} U_1 \cap U_2 &\cong j_1(C_1 \setminus B_1) \cup j_2(C_2 \setminus B_2) \\ &\cong \mathbb{S}^{n-1} \times (0, 1) \\ &\cong \mathbb{R}^n \setminus \{0\} . \end{aligned}$$

□

Theorem 93. *[Problem 4-20] Define a topology on $\mathbb{Z}$ by declaring a set A to be open if and only if $n \in A$ implies $-n \in A$. Then $\mathbb{Z}$ with this topology is second countable and limit point compact but not compact.*

Proof. Let $B_i = \{-i, i\}$. Then $\mathcal{B} = \{B_0, B_1, \ldots\}$ is a countable basis for $\mathbb{Z}$. Now let U be a subset of $\mathbb{Z}$ with at least two elements, and choose some nonzero $n \in U$. Then $-n$ is a limit point of U, since any neighborhood of $-n$ must contain n. In particular, $\mathbb{Z}$ is limit point compact. However, the open cover $\mathcal{B}$ of $\mathbb{Z}$ has no finite subcover, so $\mathbb{Z}$ is not compact. □

Theorem 94. *[Problem 4-21] Let V be a finite-dimensional real vector space. A* norm *on V is a real-valued function on V, written $v \mapsto |v|$, satisfying*

1. *Positivity: $|v| \geq 0$, and $|v| = 0$ if and only if $v = 0$.*
2. *Homogeneity: $|cv| = |c|\,|v|$ for any $c \in \mathbb{R}$ and $v \in V$.*
3. *Triangle inequality: $|v + w| \leq |v| + |w|$.*

A norm determines a metric by $d(v, w) = |v - w|$. In fact, all norms determine the same topology on V.

Proof. Since V is finite-dimensional, it is isomorphic to $\mathbb{R}^n$. For all $v, w \in V$ we have $|v| \leq |w| + |v - w|$ and $|w| \leq |v| + |v - w|$, so

$$||v| - |w|| \leq |v - w| .$$

This immediately shows that any norm on V is continuous. Let $|\cdot|_1$ and $|\cdot|_2$ be two norms on V, and write V_i for V equipped with the metric induced by $|\cdot|_i$. Consider the unit sphere $S_1 = \{v \in V : |v|_1 = 1\}$, which is compact. The image of S_1 under $|\cdot|_2$ is therefore compact, and there exists some $x \in S_1$ such that $|v|_2 \leq |x|_2$ for all $v \in S_1$.

Let B_2 be an open ball in V_2 of radius r around a point $x \in V$ and let B_1 be an open ball in V_1 of radius $r / |x|_2$ around x. We want to show that $B_1 \subseteq B_2$. Let $v \in B_1$ with

$v \neq x$ so that $0 < |v-x|_1 < r/|x|_2$. Then

$$|v-x|_2 = |v-x|_1 \left| \frac{v-x}{|v-x|_1} \right|_2 < \frac{r}{|x|_2} \left| \frac{v-x}{|v-x|_1} \right|_2 \leq r$$

since $(v-x)/|v-x|_1 \in S_1$. This shows that $v \in B_2$. A similar argument shows that every open ball in V_1 around a point $x \in V$ contains an open ball in V_2 around x. By Theorem 23, V_1 and V_2 have the same topology. □

Theorem 95. *[Problem 4-23] Let X be a locally compact Hausdorff space. The **one-point compactification** of X is the topological space X^* defined as follows. Let ∞ be some object not in X, and let $X^* = X \amalg \{\infty\}$ with the following topology:*

$$\begin{aligned} \mathcal{T} &= \{\textit{open subsets of } X\} \\ &\quad \cup \{U \subseteq X^* : X^* \setminus U \textit{ is a compact subset of } X\}. \end{aligned}$$

(1) *$\mathcal{T}$ is a topology.*
(2) *X^* is a compact Hausdorff space.*
(3) *X is dense in X^*if and only if X^* is noncompact.*
(4) *X is open and and has the subspace topology.*
(5) *A sequence of points in X diverges to infinity if and only if it converges to ∞ in X^*.*

Proof. Clearly $\emptyset, X^* \in \mathcal{T}$ since $X^* \setminus X^* = \emptyset$ is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be a subset of $\mathcal{T}$. Write $A = A_1 \cup A_2$ where for each $\alpha \in A_1$ the set U_α is open in X, and for each $\alpha \in A_2$ the set $X^* \setminus U_\alpha$ is a compact subset of X (and therefore $\infty \in U_\alpha$). If A_2 is empty then $\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A_1} U_\alpha$ is open in X since each U_α is open in X. If A_2 is not empty then

$$\begin{aligned} X^* \setminus \bigcup_{\alpha \in A} U_\alpha &= \left(\bigcap_{\alpha \in A_1} X^* \setminus U_\alpha \right) \cap \left(\bigcap_{\alpha \in A_2} X^* \setminus U_\alpha \right) \\ &= \left(\bigcap_{\alpha \in A_1} X \setminus U_\alpha \right) \cap \left(\bigcap_{\alpha \in A_2} X^* \setminus U_\alpha \right) \end{aligned}$$

is a compact subset of X since each $X \setminus U_\alpha$ is closed for $\alpha \in A_1$ and each $X^* \setminus U_\alpha$ is compact for $\alpha \in A_2$. Therefore $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in A}$ be a finite subset of $\mathcal{T}$ and partition A into the subsets A_1 and A_2 as above. If A_1 is empty then

$$X^* \setminus \bigcap_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} X^* \setminus U_\alpha$$

is compact since A is finite, so $\bigcap_{\alpha \in A} U_\alpha \in \mathcal{T}$. If A_1 is not empty then

$$\bigcap_{\alpha \in A} U_\alpha = \left(\bigcap_{\alpha \in A_1} U_\alpha \right) \cap \left(\bigcap_{\alpha \in A_2} X^* \setminus (X^* \setminus U_\alpha) \right)$$

$$= \left(\bigcap_{\alpha \in A_1} U_\alpha \right) \cap \left(\bigcap_{\alpha \in A_2} X \setminus (X^* \setminus U_\alpha) \right)$$

is open since for each $\alpha \in A_2$ the set $X^* \setminus U_\alpha$ is compact and therefore closed since X is Hausdorff. This proves that $\mathcal{T}$ is a topology.

We now prove that X^* is compact Hausdorff. Since X is already Hausdorff, it suffices to check that ∞ and any $x \in X$ can be separated by neighborhoods. Let E be a precompact neighborhood of x and let $F = X^* \setminus \overline{E}$. Then F is a neighborhood of ∞ and $E \cap F = \emptyset$, which shows that X^* is Hausdorff. Now let $\mathcal{U}$ be an open cover of X^*. Choose some $U \in \mathcal{U}$ containing ∞ so that $X^* \setminus U$ is a compact subset of X. Then there exists some finite subcover $\{U_1, \ldots, U_k\}$ of $X^* \setminus U$, and $\{U_1, \ldots, U_k, U\}$ is a finite subcover of X^*.

Suppose that X is noncompact and let U be a nonempty open set in X^*. If U contains no points from X then $U = \{\infty\}$. Then $X^* \setminus U = X$ is compact, which contradicts our assumption that X is noncompact. This proves that X is dense in X^*. Conversely, if X is compact then $\{\infty\}$ is open and therefore X cannot be dense in X^*.

Let $\{q_n\}$ be a sequence of points in X. Suppose that $\{q_n\}$ diverges to infinity and let U be a neighborhood of ∞. Then $X^* \setminus U$ is compact and contains finitely many values of q_n, so U contains infinitely many values of q_n. Therefore $q_n \to \infty$. Conversely, if $q_n \to \infty$ and $K \subseteq X$ is compact then $X^* \setminus K$ is a neighborhood of ∞, so $X^* \setminus K$ contains all but finitely many values of q_n and K contains finitely many values of q_n. ☐

Theorem 96. *[Problem 4-25] Let $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ be stereographic projection, as defined in Example 3.6. Then σ extends to a homeomorphism of $\mathbb{S}^n$ with the one-point compactification of $\mathbb{R}^n$.*

Proof. Let $S^* = (\mathbb{S}^n \setminus \{N\})^*$. By Theorem 98, σ extends to a continuous map $\sigma^* : S^* \to (\mathbb{R}^n)^*$ with $\sigma^*(\infty) = \infty$. Furthermore, $(\sigma^*)^{-1}$ is also continuous since σ^{-1} extends to a continuous map taking ∞ to ∞. Therefore σ^* is a homeomorphism. From Theorem 97 we have a homeomorphism $\varphi : \mathbb{S}^n \to S^*$, and $\sigma^* \circ \varphi$ is the required homeomorphism. ☐

Theorem 97. *[Problem 4-26] Let M be a compact manifold of positive dimension, and let $p \in M$. Then M is homeomorphic to the one-point compactification of $M \setminus \{p\}$.*

Proof. Let $\widehat{M} = (M \setminus \{p\})^*$ and define $\varphi : M \to \widehat{M}$ by

$$\varphi(x) = \begin{cases} x & \text{if } x \neq p, \\ \infty & \text{if } x = p. \end{cases}$$

Let U be open in $\widehat{M}$. If $\infty \notin U$ then $\varphi^{-1}(U)$ is open. If $\infty \in U$ then $\widehat{M} \setminus U$ is compact, so

$$\varphi^{-1}(U) = M \setminus \varphi^{-1}(\widehat{M} \setminus U) = M \setminus (\widehat{M} \setminus U)$$

is open since $\widehat{M} \setminus U$ is closed. Therefore φ is continuous. By Lemma 4.50, φ is a homeomorphism. ☐

Theorem 98. *[Problem 4-27] If X and Y are noncompact, locally compact Hausdorff spaces, then a continuous map $f : X \to Y$ extends to a continuous map $f^* : X^* \to Y^*$ taking ∞ to ∞ if and only if it is proper.*

Proof. Suppose that f is proper. Set $f^*(x) = f(x)$ for all $x \in X$ and $f^*(\infty) = \infty$. Let U be open in Y^*. If $\infty \notin U$ then $(f^*)^{-1}(U)$ is open since f is continuous. If $\infty \in U$ then $Y^* \setminus U = Y \setminus U$ is compact, and $(f^*)^{-1}(Y^* \setminus U) = f^{-1}(Y \setminus U)$ where $f^{-1}(Y \setminus U)$ is compact since f is proper. Then

$$\begin{aligned}(f^*)^{-1}(U) &= X^* \setminus [(f^*)^{-1}(Y^* \setminus U)] \\ &= X \setminus f^{-1}(Y \setminus U)\end{aligned}$$

is open since $f^{-1}(Y \setminus U)$ is closed. Conversely, suppose that f extends to a continuous map f^*. Let $E \subseteq Y$ be compact. Then $Y^* \setminus E$ is open, so

$$(f^*)^{-1}(Y^* \setminus E) = X^* \setminus (f^*)^{-1}(E)$$

is open. Since $\infty \in X^* \setminus (f^*)^{-1}(E)$ we have that $(f^*)^{-1}(E) = f^{-1}(E)$ is compact. ☐

Theorem 99. *[Problem 4-30] Suppose X is a topological space and $\{A_\alpha\}$ is a locally finite closed cover of X. If for each $\alpha \in A$ we are given a continuous map $f_\alpha : X_\alpha \to Y$ such that $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$ for all α and β, then there exists a unique continuous map $f : X \to Y$ whose restriction to each X_α is f_α.*

Proof. For each $x \in X$ we have $x \in X_\alpha$ for some α, so we can set $f(x) = f_\alpha(x)$. This makes f a well-defined map since $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$ for all α and β. Let $x \in X$ and choose a neighborhood U of x that intersects with finitely many elements $A_{\alpha_1}, \ldots, A_{\alpha_k} \in \{A_\alpha\}$. Let E be a closed subset of $f(U)$ so that $E = F \cap f(U)$ for some F closed in Y. Then

$$\begin{aligned}(f|_U)^{-1}(E) &= f^{-1}(E) \cap U \\ &= f^{-1}(F) \cap U \\ &= \bigcup_{i=1}^{k} f_{\alpha_i}^{-1}(F) \cap U\end{aligned}$$

which is closed in U since each $f_{\alpha_i}^{-1}(F)$ is closed. This shows that every point of X has a neighborhood U on which $f|_U$ is continuous. By Proposition 2.19, f is continuous. ☐

Theorem 100. *[Problem 4-33] Suppose X is a topological space with the property that for every open cover of X, there exists a partition of unity subordinate to it. Then X is paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover X. By our hypothesis, there exists a partition of unity $\{\psi_\alpha\}_{\alpha \in A}$ subordinate to $\mathcal{U}$. For each α, let $V_\alpha = \psi_\alpha^{-1}((0,1])$. We want to show that $\{V_\alpha\}$ is a locally finite open refinement of $\mathcal{U}$. Each V_α is open in X since $(0,1]$ is always open in $\psi_\alpha(X)$. Furthermore, $V_\alpha \subseteq \operatorname{supp} \psi_\alpha \subseteq U_\alpha$, so $\{V_\alpha\}$ is a refinement of $\mathcal{U}$. If $x \in X$ then there is a neighborhood N of x that intersects with a finite number of sets in $\{\operatorname{supp} \psi_\alpha\}$. In particular, N intersects with a finite number of sets in $\{V_\alpha\}$ since $V_\alpha \subseteq \operatorname{supp} \psi_\alpha$ for every α. □

Chapter 5. Cell Complexes

Theorem 101. *[Exericse 5.3] Suppose X is a topological space whose topology is coherent with a family $\mathcal{B}$ of subspaces.*

1. *If Y is another topological space, then a map $f : X \to Y$ is continuous if and only if $f|_B$ is continuous for every $B \in \mathcal{B}$.*
2. *The map $\coprod_{B \in \mathcal{B}} B \to X$ induced by inclusion of each set $B \hookrightarrow X$ is a quotient map.*

Proof. If f is continuous then it is clear that every $f|_B$ is continuous. Suppose that every $f|_B$ is continuous and let U be open in Y. Then for every $B \in \mathcal{B}$ the set $(f|_B)^{-1}(U) = f^{-1}(U) \cap B$ is open in B, so $f^{-1}(U)$ is open since X is coherent with $\mathcal{B}$. Part (2) follows directly from the definition of coherent. □

Theorem 102. *[Exercise 5.19] Suppose X is an n-dimensional CW complex with $n \geq 1$, and e_0 is any n-cell of X. Then $X \setminus e_0$ is a subcomplex, and X is homeomorphic to an adjunction space obtained from $X \setminus e_0$ by attaching a single n-cell.*

Proof. If e is a cell of $X \setminus e_0$ then $\overline{e} \setminus e$ is contained in X_{n-1}, and in particular $\overline{e} \cap e_0 = \emptyset$. Therefore $X \setminus e_0$ is a subcomplex. Let $\phi : D_0 \to X$ be a characteristic map for e_0 and form the adjunction space $(X \setminus e_0) \cup_\phi D_0$. Define $\psi : (X \setminus e_0) \amalg D_0 \to X$ as being equal to inclusion on $X \setminus e_0$ and to ϕ on D_0. Then ψ makes the same identifications as the quotient map defining the adjunction space, so it remains to show that ψ is a quotient map. The argument is identical to that of Proposition 5.18. □

Theorem 103. *[Exercise 5.34] If K is a Euclidean simplicial complex, then the collection $\widehat{K}$ consisting of the interiors of the simplices of K is a regular CW decomposition of $|K|$.*

Proof. First note that the sets in $\widehat{K}$ are disjoint, for the intersection of two simplices is either empty or a face of each, and if a point is in the interior of a simplex σ, it cannot be in a face of σ. This shows that $\widehat{K}$ is a partition of $|K|$. Proposition 5.32 shows that for each $\sigma \in K$ we have a homeomorphism $\phi : \triangle_k \to \sigma$ where $\triangle_k$ is the standard k-simplex. Since ϕ restricts to a homeomorphism from $\operatorname{Int}\triangle_k$ to the interior of σ and $\phi(\partial\triangle_k)$ maps into the boundary of σ, we can take ϕ as a characteristic map for the interior of σ. Furthermore, since K is locally finite, $\widehat{K}$ must also be locally finite. By Proposition 5.4, $\widehat{K}$ is a regular CW decomposition of $|K|$. □

Theorem 104. *[Exercise 5.40] Let K and L be simplicial complexes. Suppose $f_0 : K_0 \to L_0$ is any map with the property that whenever $\{v_0, \dots, v_k\}$ are the vertices of a simplex of K, $\{f_0(v_0), \dots, f_0(v_k)\}$ are the vertices of a simplex of L (possibly with repetitions). Then there is a unique simplicial map $f : |K| \to |L|$ whose vertex map is f_0. It is a simplicial isomorphism if and only if f_0 is a bijection satisfying the following additional condition: $\{v_0, \dots, v_k\}$ are the vertices of a simplex of K if and only if $\{f_0(v_0), \dots, f_0(v_k)\}$ are the vertices of a simplex of L.*

Proof. Suppose K is in $\mathbb{R}^n$ and L is in $\mathbb{R}^m$. Let $\{V_\alpha\}_{\alpha\in A}$ be the collection of all subsets K_0 that define a vertex of a simplex of K. For each $V_\alpha = \{v_0, \dots, v_k\}$ we know that $\{f_0(v_0), \dots, f_0(v_k)\}$ are the vertices of a simplex of L, so by Proposition 5.38 there exists a unique map $f_\alpha : \sigma \to \mathbb{R}^m$ that is the restriction of an affine map, takes v_i to $f_0(v_i)$ for each i, and takes σ onto a simplex of L. Since the maps f_α agree on the intersection of their domains and K is locally finite, Theorem 99 shows that there is a unique (simplicial) map $f : |K| \to |L|$. If in addition f_0 is a bijection satisfying the given condition then a similar process gives an inverse to f. □

Theorem 105. *[Problem 5-1] Suppose D and D' are closed cells (not necessarily of the same dimension).*

1. *Every continuous map $f : \partial D \to \partial D'$ extends to a continuous map $f : D \to D'$, with $F(\operatorname{Int} D) \subseteq \operatorname{Int} D'$.*
2. *Given points $p \in \operatorname{Int} D$ and $p' \in \operatorname{Int} D'$, F can be chosen to take p to p'.*
3. *If f is a homeomorphism, then F can also be chosen to be a homeomorphism.*

Proof. We have homeomorphisms $\varphi : \overline{\mathbb{B}^n} \to D$ and $\psi : \overline{\mathbb{B}^m} \to D'$ such that $\varphi(\mathbb{S}^{n-1}) = \partial D$ and $\psi(\mathbb{S}^{m-1}) = \partial D'$. Define $\widetilde{f} : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1}$ by $\widetilde{f} = \psi^{-1} \circ f \circ \varphi$. Define $\widetilde{F} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^m}$ by $\widetilde{F}(0) = 0$ and

$$(*) \qquad \widetilde{F}(x) = \|x\|\, \widetilde{f}\left(\frac{x}{\|x\|}\right)$$

for $x \neq 0$. Then $\widetilde{F}$ is a continuous extension of $\widetilde{f}$, and $F = \psi \circ \widetilde{F} \circ \varphi^{-1}$ is a continuous extension of f. If f is a homeomorphism then $\widetilde{F}$ is easily seen to be a bijection, and by

Lemma 4.25, $\widetilde{F}$ is a homeomorphism. For part (2), Proposition 5.1 shows that there exist homeomorphisms $G : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ and $G' : \overline{\mathbb{B}^m} \to \overline{\mathbb{B}^m}$ such that $G(0) = \varphi^{-1}(p)$ and $G'(0) = \psi^{-1}(p')$. Modify the previous definitions by setting $\widetilde{f} = (G')^{-1} \circ \psi^{-1} \circ f \circ \varphi \circ G$ and $F = \psi \circ G' \circ \widetilde{F} \circ G^{-1} \circ \varphi^{-1}$. Then

$$\begin{aligned} F(p) &= (\psi \circ G' \circ \widetilde{F} \circ G^{-1})(\varphi^{-1}(p)) \\ &= (\psi \circ G' \circ \widetilde{F})(0) \\ &= (\psi \circ G')(0) \\ &= p' \end{aligned}$$

as desired. ☐

Theorem 106. *[Problem 5-2] Suppose D is a closed n-cell, $n \geq 1$.*

(1) *Given any point $p \in \operatorname{Int} D$, there is a continuous function $F : D \to [0,1]$ such that $F^{-1}(\{1\}) = \partial D$ and $F^{-1}(\{0\}) = \{p\}$.*
(2) *Any continuous function $f : \partial D \to [0,1]$ extends to a continuous function $F : D \to [0,1]$ that is strictly positive in $\operatorname{Int} D$.*

Proof. For (1), apply Theorem 105 to the map $f : \partial D \to [-1,1]$ satisfying $f(x) = 1$ for all $x \in D$, obtaining a map $F : D \to [-1,1]$ with $F(p) = 0$. From (*) it is easy to see that $F^{-1}(\{1\}) = \partial D$ and $F^{-1}(\{0\}) = \{p\}$. For (2), repeat the proof of Theorem 105 but set $\widetilde{F}(0) = 1$ and

$$\widetilde{F}(x) = 1 - \|x\| \, \widetilde{f}\left(\frac{x}{\|x\|}\right)$$

for $x \neq 0$. ☐

Theorem 107. *Let X be a connected topological space and let $\sim$ be an equivalence relation on X. If every $x \in X$ has a neighborhood U such that $p \sim q$ for every $p, q \in U$, then $p \sim q$ for every $p, q \in X$.*

Proof. Let $p \in X$ and let $S = \{q \in X : p \sim q\}$. If $q \in S$ then there is a neighborhood U of q such that $q_1 \sim q_2$ for every $q_1, q_2 \in U$. In particular, for every $r \in U$ we have $p \sim q$ and $q \sim r$ which implies that $p \sim r$, and $U \subseteq S$. This shows that S is open. If $q \in X \setminus S$ then there is a neighborhood U of q such that $q_1 \sim q_2$ for every $q_1, q_2 \in U$. If $p \sim r$ for some $r \in U$ then $p \sim q$ since $q \sim r$, which contradicts the fact that $q \in X \setminus S$. Therefore $U \subseteq X \setminus S$, which shows that S is closed. Since X is connected, $S = X$. ☐

Theorem 108. *[Problem 5-3]*

(1) *Given any two points $p, q \in \mathbb{B}^n$, there is a homeomorphism $\varphi : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ such that $\varphi(p) = q$ and $\varphi|_{\partial \mathbb{B}^n} = \operatorname{Id}_{\partial \mathbb{B}^n}$.*

(2) *For any topological manifold X, every point of X has a neighborhood U with the property that for any $p, q \in U$, there is a homeomorphism from X to itself taking p to q.*
(3) *Every connected topological manifold is topologically homogeneous.*

Proof. Part (1) follows from applying Theorem 105 to the map $\mathrm{Id}_{\partial\mathbb{B}^n}$. Let $x \in X$ and choose a regular coordinate ball B around x. If $p, q \in B$ then it follows from part (1) that there exists a homeomorphism $\varphi : \overline{B} \to \overline{B}$ such that $\varphi(p) = q$ and $\varphi|_{\partial B} = \mathrm{Id}_{\partial B}$. Since $\{\overline{B}, X \setminus B\}$ is a closed cover of X, the gluing lemma shows that there is a (unique) homeomorphism $\psi : X \to X$ taking p to q satisfying $\psi|_{X\setminus B} = \mathrm{Id}_{X\setminus B}$. Part (3) follows by applying Theorem 107 to the equivalence relation defined by $p \sim q$ if and only if there exists a homeomorphism from X to itself taking p to q. □

Lemma 109. *Let X be a Hausdorff space. If $p_1, \dots, p_n$ are distinct points in X, then there exist neighborhoods $U_1, \dots, U_n$ with $p_i \in U_i$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$.*

Proof. We use induction on n. If $n = 2$ then the statement is clearly true. Assume that the statement is true for n distinct points and let $p_1, \dots, p_{n+1}$ be distinct points in X. There exist neighborhoods $U_1, \dots, U_n$ with $p_i \in U_i$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$. For each $1 \leq i \leq n$, choose a neighborhoods E_i of p_i and F_i of p_{n+1} such that $E_i \cap F_i = \emptyset$. Then $U_1 \cap E_1, \dots, U_n \cap E_n, F_1 \cap \cdots \cap F_n$ are the desired neighborhoods of $p_1, \dots, p_{n+1}$ respectively. □

Theorem 110. *[Problem 5-4] If M is a connected topological manifold with* $\dim M > 1$ *and $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ are two ordered k-tuples of distinct points in M, then there is a homeomorphism $F : M \to M$ such that $F(p_i) = q_i$ for $i = 1, \dots, k$.*

Proof. Choose neighborhoods $U_1, \dots, U_k$ with $p_i \in U_i$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$ and similarly choose neighborhoods $V_1, \dots, V_k$ with $q_i \in V_i$. Then choose regular coordinate balls $E_1, \dots, E_k$ around $p_1, \dots, p_k$ and regular coordinate balls $F_1, \dots, F_k$ around $q_1, \dots, q_k$. For each $1 \leq j \leq k$ the manifold $M_j = M \setminus \bigcup_{i \neq j} \overline{E_i} \cup \overline{F_i}$ is connected, so Theorem 108 shows that there exists a homeomorphism $\varphi_j : M_j \to M_j$ that takes p_j to q_j. By the gluing lemma, we can extend φ_j to a homeomorphism $\psi_j : M \to M$ such that $\psi_j|_{\overline{E_i} \cup \overline{F_i}}$ is the identity for every $i \neq j$. Then $\psi = \psi_1 \circ \cdots \circ \psi_k$ is the desired homeomorphism taking p_j to q_j for $j = 1, \dots, k$. □

Theorem 111. *[Problem 5-5] Suppose X is a topological space and $\{X_\alpha\}$ is a family of subspaces whose union is X. The topology of X is coherent with the subspaces $\{X_\alpha\}$ if and only if it is the finest topology on X for which all of the inclusion maps $X_\alpha \hookrightarrow X$ are continuous.*

Proof. Suppose that the topology $\mathcal{T}$ of X is coherent with $\{X_\alpha\}$ and let $\mathcal{T}'$ be some topology for which the inclusion maps $i_\alpha : X_\alpha \hookrightarrow X$ are continuous. If $U \in \mathcal{T}'$ then every $i_\alpha^{-1}(U) = U \cap X_\alpha$ is open, so U is open in $\mathcal{T}$ since $\mathcal{T}$ is coherent with $\{X_\alpha\}$. This shows that $\mathcal{T}' \subseteq \mathcal{T}$. Conversely, suppose that $\mathcal{T}$ is the finest topology for which the inclusion maps i_α are continuous. Let U be a subset of X such that $U \cap X_\alpha \in \mathcal{T}$ for all α. If U is not open in $\mathcal{T}$ then by defining a new topology $\mathcal{T}' \supseteq \mathcal{T}$ such that $U \in \mathcal{T}'$, we obtain a finer topology such that every i_α is continuous. This is a contradiction, so U must be open in $\mathcal{T}$. □

Theorem 112. *[Problem 5-6] Suppose X is a topological space. The topology of X is coherent with each of the following collections of subspaces of X:*

(1) *Any open cover of X.*
(2) *Any locally finite closed cover of X.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X and let U be a subset of X such that $U \cap U_\alpha$ is open for every $\alpha \in A$. Then

$$U = U \cap X = U \cap \bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\in A} U \cap U_\alpha$$

is open, which proves (1). Now let $\{E_\alpha\}_{\alpha\in A}$ be a locally finite closed cover of X and let E be a subset of X such that $E \cap E_\alpha$ is closed for every $\alpha \in A$. Let x be a limit point of E; we want to show that $x \in E$. Since $\{E_\alpha\}$ is locally finite, there exists a neighborhood U of x that intersects with finitely many elements $E_{\alpha_1}, \ldots, E_{\alpha_n} \in \{E_\alpha\}$. If V is a neighborhood of x then $U \cap V$ is also a neighborhood of x, so there exists some $y \in U \cap V$ not equal to x such that $y \in E$. But then $y \in E_{\alpha_i}$ for some i, which shows that x is a limit point of $E \cap \bigcup_{i=1}^n E_{\alpha_i}$. Since

$$E \cap \bigcup_{i=1}^{n} E_{\alpha_i} = \bigcup_{i=1}^{n} E \cap E_{\alpha_i}$$

is closed, $x \in E \cap \bigcup_{i=1}^n E_{\alpha_i} \subseteq E$. This proves (2). □

Theorem 113. *[Problem 5-7] Suppose X is a topological space whose topology is coherent with a collection $\{X_\alpha\}_{\alpha\in A}$ of subspaces of X, and for each $\alpha \in A$ we are given a continuous map $f_\alpha : X_\alpha \to Y$ such that $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ for all α and β. Then there exists a unique continuous map $f : X \to Y$ whose restriction to each X_α is f_α (cf. Theorem 99).*

Proof. For each $x \in X$ we have $x \in X_\alpha$ for some α, so we can set $f(x) = f_\alpha(x)$. This makes f a well-defined map since $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ for all α and β. Let U be open in Y. Then for every $\alpha \in A$ we have that $f_\alpha^{-1}(U) = f^{-1}(U) \cap X_\alpha$ is open in X_α, so $f^{-1}(U)$ is open since X is coherent with $\{X_\alpha\}$. □

Theorem 114. *[Problem 5-8] If X is any CW complex, the topology of X is coherent with the collection of subspaces $\{X_n : n \geq 0\}$.*

Proof. Let U be a subset of X and suppose that $U \cap X_n$ is closed in X_n for every $n \geq 0$. If e is an n-cell in X then $\overline{e} \cap U \cap X_n$ is closed in $\overline{e}$ since $\overline{e}$ is closed in X_n. By condition (W), U is closed in X. □

Theorem 115. *[Problem 5-10] Every CW complex is compactly generated.*

Proof. Let X be a CW complex and let U be a subset of X such that $U \cap K$ is closed for every compact set $K \subseteq X$. Then every $U \cap \overline{e}$ is closed, so U is closed by condition (W). □

Theorem 116. *[Problem 5-11] A CW complex is locally compact if and only if it is locally finite.*

Proof. Let X be a CW complex. Suppose that X is locally finite and let $x \in X$. Since the collection $\{\overline{e} : e \in \mathcal{E}\}$ is locally finite, there exists a neighborhood U of x that intersects with finitely many elements $\overline{e_1}, \ldots, \overline{e_n}$. Then U is a precompact neighborhood of x, since $\overline{U}$ is a closed subset of the compact set $\overline{e_1} \cup \cdots \cup \overline{e_n}$. Conversely, suppose that X is locally compact, let $x \in X$ and let U be a precompact neighborhood of x. By Theorem 5.14, $\overline{U}$ is contained in a finite subcomplex, so U intersects finitely many cells of X. □

Theorem 117. *[Problem 5-12] Let $\mathbb{P}^n$ be n-dimensional projective space. The usual inclusion $\mathbb{R}^{k+1} \subseteq \mathbb{R}^{n+1}$ for $k < n$ allows us to consider $\mathbb{P}^k$ as a subspace of $\mathbb{P}^n$. Then $\mathbb{P}^n$ has a CW decomposition with one cell in each dimension $0, \ldots, n$ such that the k-skeleton is $\mathbb{P}^k$ for $0 < k < n$.*

Proof. We use induction on n. The result is clearly true for $n = 0$, so assume that the result is true for $\mathbb{P}^{n-1}$ and consider $\mathbb{P}^n$. Define the map

$$\begin{aligned} F : \overline{\mathbb{B}^n} &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (x_1, \ldots, x_n) &\mapsto \left(x_1, \ldots, x_n, \sqrt{1 - |x_1|^2 - \cdots - |x_n|^2} \right) \end{aligned}$$

and let $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map making two nonzero points $x, y \in \mathbb{R}^{n+1}$ equivalent if $x = \lambda y$ for some $\lambda \in \mathbb{R}$. We can write $\mathbb{P}^n$ as the disjoint union of $\mathbb{P}^{n-1}$ and the set $Q = \{[p_1, \ldots, p_n] \in \mathbb{P}^n : p_n \neq 0\}$. It is clear that $F(\partial \mathbb{B}^n) \subseteq \mathbb{R}^n \times \{0\}$, so $(q \circ F)(\partial \mathbb{B}^n) \subseteq \mathbb{P}^{n-1}$; it is also clear that $(q \circ F)(\mathbb{B}^n) = Q$ since $F(\mathbb{B}^n)$ is the upper hemisphere of $\mathbb{S}^{n-1}$. Finally, $(F|_{\mathbb{B}^n})^{-1}$ is the map that discards the last coordinate, which is continuous. This shows that Q is an n-cell with characteristic map $q \circ F$, and that $\mathbb{P}^n$ is a CW complex. □

Theorem 118. *[Problem 5-13] Let $\mathbb{CP}^n$ be n-dimensional complex projective space. Then $\mathbb{CP}^n$ has a CW decomposition with one cell in each even dimension $0, 2, \ldots, 2n$ such that the 2k-skeleton is $\mathbb{CP}^k$ for $0 < k < n$.*

Proof. Proceed as in Theorem 117, replacing $\mathbb{B}^n$ with the complex unit ball

$$B_n = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sqrt{|z_1|^2 + \cdots + |z_n|^2} = 1 \right\}$$

which has dimension $2n$. □

Theorem 119. *[Problem 5-14] Every nonempty compact convex subset $D \subseteq \mathbb{R}^n$ is a closed cell of some dimension.*

Proof. Let $x + S$ be an affine subspace of minimal dimension k containing D and let $A = [v_0, \dots, v_\ell]$ be a simplex of maximal dimension contained in D. Suppose that $\ell < k$. There exists a point $y \in D$ affinely independent from $v_0, \dots, v_\ell$, for otherwise D would be contained in the affine subspace spanned by A, which has dimension less than k. Since D is convex, it contains the simplex $[v_0, \dots, v_\ell, y]$, which has dimension $\ell + 1$. But this contradicts the fact that A is a simplex of maximal dimension contained in D. Therefore $\ell = k$, and considering D as a subset of $x + S$, the interior of D is nonempty. Now let $\varphi : S \to \mathbb{R}^k$ be an isomorphism, which is also a homeomorphism. Applying Proposition 5.1 to $\varphi(D)$ shows that D is a closed k-cell. □

Chapter 6. Compact Surfaces

Theorem 120. *[Exercise 6.11] Each elementary transformation of a polygonal presentation produces a topologically equivalent presentation.*

Proof. We only prove the result for subdividing and reflecting. Let $\mathcal{P} = \langle S \mid W_1, \dots, W_k \rangle$ be a polygonal presentation and let $\mathcal{P}' = \langle S, e \mid W_1', \dots, W_k' \rangle$ be the presentation formed by replacing every occurrence of a by ae and every occurrence of a^{-1} by $e^{-1}a^{-1}$, taking each word W_i to W_i'. First assume $\mathcal{P}$ has words of length 3 or more. Let $P_1, \dots, P_k$ be polygonal regions for $\mathcal{P}$, let $P_1', \dots, P_k'$ be the polygonal regions for $\mathcal{P}$, and let $\pi : \coprod_{i=1}^k P_i \to M$ and $\pi' : \coprod_{i=1}^k P_i' \to M'$ be the quotient maps. Let $f : \coprod_{i=1}^k P_i \to \coprod_{i=1}^k P_i'$ be a map that takes edges labeled with a to the two corresponding edges labeled a and e (such that the preimage of a has length $1/2$), and similarly for a^{-1}. This map can be chosen to be a homeomorphism by Theorem 105. Since $\pi' \circ f$ makes the same identifications as π, M and M' are homeomorphic. If $\mathcal{P}$ has a single word of length 2, it is easy to check that the individual cases are homeomorphic to the sphere or projective plane (as is noted in Example 6.9). For reflection, define $f : \coprod_{i=1}^k P_i \to \coprod_{i=1}^k P_i'$ by sending each edge to itself, but with the opposite orientation. □

Example 121. [Problem 6-1] Show that a connected sum of one or more projective planes contains a subspace that is homeomorphic to the Möbius band.

Since $\mathbb{P}^2\#\mathbb{P}^2$ is homeomorphic to the Klein bottle and the Klein bottle contains a copy of the Möbius band.

Example 122. [Problem 6-2] Note that both a disk and a Möbius band are manifolds with boundary, and both boundaries are homeomorphic to $\mathbb{S}^1$. Show that it is possible to obtain a space homeomorphic to a projective plane by attaching a disk to a Möbius band along their boundaries.

We have the presentation $\langle a,b,c,d,e \mid abcdec\rangle$ for a Möbius band and the presentation $\langle a,b,c,d \mid abe^{-1}d^{-1}\rangle$ for a (closed) disk with its edges identified with the boundary of the Möbius band. Attaching the disk to the Möbius band gives the presentation

$$\begin{aligned}\left\langle a,b,c,d,e \mid abe^{-1}d^{-1}, abcdec\right\rangle &\approx \left\langle a,b,c,d,e \mid abe^{-1}d^{-1}, decabc\right\rangle \\ &\approx \langle a,b,c \mid abcabc\rangle \\ &\approx \langle a \mid aa\rangle,\end{aligned}$$

which is the projective plane.

Example 123. [Problem 6-3] Show that the Klein bottle is homeomorphic to a quotient obtained by attaching two Möbius bands together along their boundaries.

This corresponds to the presentation

$$\begin{aligned}\left\langle a,b,c,d \mid abcb, a^{-1}dc^{-1}d\right\rangle &\approx \left\langle a,b,c,d \mid bcba, a^{-1}dc^{-1}d\right\rangle \\ &\approx \left\langle b,c,d \mid bcbdc^{-1}d\right\rangle \\ &\approx \left\langle b,c,d,e \mid bce, e^{-1}bdc^{-1}d\right\rangle \\ &\approx \left\langle b,c,d,e \mid ebc, c^{-1}de^{-1}bd\right\rangle \\ &\approx \left\langle b,d,e \mid ebde^{-1}bd\right\rangle \\ &\approx \left\langle e,f \mid efe^{-1}f\right\rangle \\ &\approx \left\langle e,f \mid fefe^{-1}\right\rangle,\end{aligned}$$

which is the Klein bottle.

Theorem 124. *[Problem 6-5] Every compact 2-manifold with boundary is homeomorphic to a compact 2-manifold with finitely many open cells removed.*

Proof. Let M be a compact 2-manifold with boundary. By Theorem 5.27, there is a homeomorphism $\varphi : \partial M \to \coprod_{i=1}^k \mathbb{S}^1$. Let M' be the compact 2-manifold formed by attaching half of a sphere to each $\varphi^{-1}(\mathbb{S}^1)$; then M is homeomorphic to M' with the interiors of the half-spheres removed. □

Example 125. [Problem 6-6] For each of the following surface presentations, compute the Euler characteristic and determine which of our standard surfaces it represents.

(1) $\langle a, b, c \mid abacb^{-1}c^{-1}\rangle$
(2) $\langle a, b, c \mid abca^{-1}b^{-1}c^{-1}\rangle$

We run the classification algorithm on each of the presentations:

$$\begin{aligned}\left\langle a, b, c \mid abacb^{-1}c^{-1}\right\rangle &\approx \left\langle a, b, c \mid cb^{-1}c^{-1}aba\right\rangle \\ &\approx \left\langle a, b, c, d \mid cb^{-1}c^{-1}ad, d^{-1}ba\right\rangle \\ &\approx \left\langle a, b, c, d \mid dcb^{-1}c^{-1}a, a^{-1}b^{-1}d\right\rangle \\ &\approx \left\langle b, c, d \mid ddcb^{-1}c^{-1}b^{-1}\right\rangle \\ &\approx \left\langle b, c, d, e \mid ddcb^{-1}e, e^{-1}c^{-1}b^{-1}\right\rangle \\ &\approx \left\langle b, c, d, e \mid eddcb^{-1}, bce\right\rangle \\ &\approx \langle c, d, e \mid ddccee\rangle,\end{aligned}$$

so (1) is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ with Euler characteristic -1;

$$\begin{aligned}\left\langle a, b, c \mid abca^{-1}b^{-1}c^{-1}\right\rangle &\approx \left\langle a, b, c \mid c^{-1}abca^{-1}b^{-1}\right\rangle \\ &\approx \left\langle a, b, c, d \mid c^{-1}ad, d^{-1}bca^{-1}b^{-1}\right\rangle \\ &\approx \left\langle a, b, c, d \mid dc^{-1}a, a^{-1}b^{-1}d^{-1}bc\right\rangle \\ &\approx \left\langle b, c, d \mid dc^{-1}b^{-1}d^{-1}bc\right\rangle \\ &\approx \left\langle b, c, d, e \mid dc^{-1}b^{-1}e, e^{-1}d^{-1}bc\right\rangle \\ &\approx \left\langle b, c, d, e \mid edc^{-1}b^{-1}, bce^{-1}d^{-1}\right\rangle \\ &\approx \left\langle d, e \mid ede^{-1}d^{-1}\right\rangle,\end{aligned}$$

so (2) is homeomorphic to the torus with Euler characteristic 0.

Chapter 7. Homotopy and the Fundamental Group

Theorem 126. *[Exercise 7.6] Let $B \subseteq \mathbb{R}^n$ be any convex set, X be any topological space, and A be any subset of X. Then any two continuous maps $f, g : X \to B$ that agree on A are homotopic relative to A.*

Proof. The straight-line homotopy between f and g is in fact a homotopy relative to A. □

Theorem 127. *[Exercise 7.8] Let X be a topological space. For any points $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q.*

Proof. This follows from Proposition 7.1, since the combined homotopy is still stationary on $\{0, 1\}$. □

Theorem 128. *[Exercise 7.14] Let X be a path-connected topological space.*

(1) *Let $f, g : I \to X$ be two paths from p to q. Then $f \sim g$ if and only if $f \cdot \overline{g} \sim c_p$.*
(2) *X is simply connected if and only if any two paths in X with the same initial and terminal points are path-homotopic.*

Proof. For (1), we have $f \sim g \Leftrightarrow f \cdot \overline{g} \sim g \cdot \overline{g} \sim c_p$. It follows that X is simply connected if and only if every element of $\pi_1(X)$ is the identity, i.e. $f \cdot \overline{g} \sim c_p$ for all paths f, g from p to q. □

Theorem 129. *[Exercise 7.15] Every convex subset of $\mathbb{R}^n$ is simply connected, and $\mathbb{R}^n$ itself is simply connected.*

Proof. This follows from Theorem 126. □

Theorem 130. *[Exercise 7.23] The path homotopy relation is preserved by composition with continuous maps. That is, if $f_0, f_1 : I \to X$ are path-homotopic and $\varphi : X \to Y$ is continuous, then $\varphi \circ f_0$ and $\varphi \circ f_1$ are path-homotopic.*

Proof. Let $H : I \times I \to X$ be a path homotopy from f_0 to f_1. Then $\varphi \circ H$ is easily seen to be a path homotopy from $\varphi \circ f_0$ to $\varphi \circ f_1$. □

Theorem 131. *[Exercise 7.27]*

(1) *A retract of a connected space is connected.*
(2) *A retract of a compact space is compact.*
(3) *A retract of a retract is a retract; that is, if $A \subseteq B \subseteq X$, A is a retract of B, and B is a retract of X, then A is a retract of X.*

Proof. Let $r : X \to A$ be a retraction; (1) and (2) follow from the fact that r is continuous. Let $r_1 : B \to A$ and $r_2 : X \to B$ be retractions. Then $r_1 \circ r_2 : X \to A$ is also a retraction, which proves (3). □

Theorem 132. *[Exercise 7.33] The circle is not a retract of the closed disk.*

Proof. The circle is not simply connected, but the closed disk is convex and therefore simply connected. By Corollary 7.29, the circle cannot be a retract of the closed disk. □

Theorem 133. *[Exercise 7.36] Homotopy equivalence is an equivalence relation on the class of all topological spaces.*

Proof. It is clear that homotopy equivalence is reflexive and symmetric. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be homotopy equivalences with homotopy inverses $\overline{\varphi} : Y \to X$ and $\overline{\psi} : Z \to Y$. Then $\overline{\varphi} \circ \overline{\psi} : Z \to X$ is a homotopy inverse for $\psi \circ \varphi : X \to Z$, since

$$\overline{\varphi} \circ \overline{\psi} \circ \psi \circ \varphi \simeq \overline{\varphi} \circ \mathrm{Id}_Y \circ \varphi \simeq \overline{\varphi} \circ \varphi \simeq \mathrm{Id}_X$$

and similarly $\psi \circ \varphi \circ \overline{\varphi} \circ \overline{\psi} \simeq \mathrm{Id}_Z$ by Theorem 130. □

Theorem 134. *[Exercise 7.42] The following are equivalent:*

(1) *X is contractible.*
(2) *X is homotopy equivalent to a one-point space.*
(3) *Each point of X is a deformation retract of X.*

Proof. (1) ⇔ (2) and (3) ⇒ (2) are obvious. If X is homotopy equivalent to a one-point space $\{p\} \subseteq X$ then X is simply connected, so there is a path $\gamma : I \to X$ from p to any point $q \in X$. Let $H : X \times I \to X$ be a deformation retraction to $\{p\}$; then

$$H'(x,t) = \begin{cases} H(x, 2t) & \text{if } 0 \le t \le 1/2, \\ \gamma(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is a deformation retraction from X to the one-point space $\{q\}$. □

Theorem 135. *[Exercise 7.58] If a coproduct exists in a category, it is unique up to an isomorphism that respects the injections.*

Proof. Let $(S, (i_\alpha))$ and $(S', (i'_\alpha))$ be two coproducts for a family of objects (X_α). By the universal property, there exist unique morphisms $f : S \to S'$ and $f' : S' \to S$ such that $i'_\alpha = f \circ i_\alpha$ and $i_\alpha = f' \circ i'_\alpha$ for every α. Then $i_\alpha = f' \circ f \circ i_\alpha$, so by uniqueness we have $f' \circ f = \mathrm{Id}_S$. Similarly, $i'_\alpha = f \circ f' \circ i'_\alpha$ implies that $f \circ f' = \mathrm{Id}_{S'}$. Therefore f is an isomorphism (respecting the injections). □

Theorem 136. *Let $X_1, \ldots, X_n$ be topological spaces.*

(1) *Let Y be any topological space and let $f, g : Y \to X_1 \times \cdots \times X_n$ be continuous maps. Then $f \simeq g$ if and only if $f_j \simeq g_j$ for every j, where $f_j = \pi_j \circ f$, $g_j = \pi_j \circ g$, and $\pi_j : X_1 \times \cdots \times X_n \to X_j$ is the canonical projection.*
(2) *Let Y be any topological space and let $f, g : X_1 \amalg \cdots \amalg X_n \to Y$ be continuous maps. Then $f \simeq g$ if and only if $f_j \simeq g_j$ for every j, where $f_j = f \circ i_j$, $g_j = g \circ i_j$, and $i_j : X_j \to X_1 \times \cdots \times X_n$ is the canonical injection.*

Proof. In both (1) and (2), if $f \simeq g$ then $f_j \simeq g_j$ for every j by Proposition 7.2. Now suppose that $H_j : Y \times I \to X_j$ are homotopies from f_j to g_j in (1). Then the map $H : Y \times I \to X_1 \times \cdots \times X_n$ given by

$$H(s,t) = (H_1(s,t), \ldots, H_n(s,t))$$

is a homotopy from f to g. Similarly, suppose that $H_j : X_j \times I \to Y$ are homotopies from f_j to g_j in (2). There exists a unique continuous map $H : (X_1 \times I) \amalg \cdots \amalg (X_n \times I) \to Y$ such that $H|_{X_j \times I} = H_j$ for every j. Let

$$\iota : (X_1 \amalg \cdots \amalg X_n) \times I \to (X_1 \times I) \amalg \cdots \amalg (X_n \times I)$$

be the identity map, which is continuous by Theorem 60. Then the map $H \circ \iota$ is a homotopy from f to g. □

Theorem 137. *[Problem 7-1] Suppose $f, g : X \to \mathbb{S}^n$ are continuous maps such that $f(x) \neq -g(x)$ for every $x \in X$. Then f and g are homotopic.*

Proof. Define $H : X \times I \to \mathbb{S}^n$ by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|};$$

we must check that the denominator is never zero. If $(1-t)f(x) + tg(x) = 0$ and $0 < t < 1$ then

$$f(x) = \frac{(1-t)f(x)}{\|(1-t)f(x)\|} = \frac{-tg(x)}{\|-tg(x)\|} = -g(x)$$

since $\|f(x)\| = \|g(x)\| = 1$. We are given that this cannot happen, so H is a homotopy from f to g. □

Theorem 138. *[Problem 7-2] Suppose X is a topological space, and g is any path in X from p to q. Let $\phi_g : \pi_1(X,p) \to \pi_1(X,q)$ denote the group isomorphism defined in Theorem 7.13.*

1. *If h is another path in X starting at q, then $\phi_{g \cdot h} = \phi_h \circ \phi_g$.*
2. *For any continuous map $\psi : X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc}
\pi_1(X,p) & \xrightarrow{\psi_*} & \pi_1(Y,\psi(p)) \\
\Big\downarrow{\scriptstyle \phi_g} & & \Big\downarrow{\scriptstyle \phi_{\psi\circ g}} \\
\pi_1(X,q) & \xrightarrow{\psi_*} & \pi_1(Y,\psi(q)).
\end{array}$$

Proof. (1) follows from the computation

$$\begin{aligned}
\phi_{g\cdot h}[f] &= [\overline{g \cdot h}] \cdot [f] \cdot [g \cdot h] \\
&= [\overline{h} \cdot \overline{g}] \cdot [f] \cdot [g \cdot h] \\
&= [\overline{h}] \cdot ([\overline{g}] \cdot [f] \cdot [g]) \cdot [h] \\
&= \phi_h \circ \phi_g.
\end{aligned}$$

For (2), we need to show that $\phi_{\psi\circ g}\circ\psi_* = \psi_*\circ\phi_g$. Since ψ_* is a homomorphism,

$$\begin{aligned}(\psi_*\circ\phi_g)[f] &= \psi_*([\overline{g}]\cdot[f]\cdot[g])\\ &= [\psi\circ\overline{g}]\cdot[\psi\circ f]\cdot[\psi\circ g]\\ &= [\overline{\psi\circ g}]\cdot[\psi\circ f]\cdot[\psi\circ g]\\ &= (\phi_{\psi\circ g}\circ\psi_*)[f].\end{aligned}$$

□

Theorem 139. *[Problem 7-3] Let X be a path-connected topological space, and let $p,q\in X$. Then $\pi_1(X,p)$ is abelian if and only if all paths from p to q give the same isomorphism of $\pi_1(X,p)$ with $\pi_1(X,q)$.*

Proof. Note that $\pi_1(X,p)$ is abelian if and only if all of its inner automorphisms are trivial. If g_1 and g_2 are paths from p to q then by 138,

$$\begin{aligned}\phi_{g_1}=\phi_{g_2} &\Leftrightarrow \phi_{g_2}^{-1}\circ\phi_{g_1}=\mathrm{Id}_{\pi_1(X,p)}\\ &\Leftrightarrow \phi_{\overline{g_2}}\circ\phi_{g_1}=\mathrm{Id}_{\pi_1(X,p)}\\ &\Leftrightarrow \phi_{g_1\cdot\overline{g_2}}=\mathrm{Id}_{\pi_1(X,p)}\,.\end{aligned}$$

Therefore it suffices to show that $\phi_{g_1\cdot\overline{g_2}}=\mathrm{Id}_{\pi_1(X,p)}$ for all paths g_1,g_2 from p to q if and only if every inner automorphism of $\pi_1(X,p)$ is trivial. One direction is immediate, for $\phi_{g_1\cdot\overline{g_2}}$ is always an inner automorphism. Conversely, if ϕ_g is an inner automorphism (where g is a loop based at p) and h is a path from p to q then $\phi_g=\phi_{(g\cdot h)\cdot\overline{h}}=\mathrm{Id}_{\pi_1(X,p)}$. □

Theorem 140. *[Problem 7-4] Let $F: I\times I\to X$ be a continuous map, and let f, g, h, and k be the paths in X defined by*

$$\begin{aligned}f(s)&=F(s,0);\\ g(s)&=F(1,s);\\ h(s)&=F(0,s);\\ k(s)&=F(s,1).\end{aligned}$$

Then $f\cdot g\sim h\cdot k$.

Proof. Define $G: I\times I\to I\times I$ by

$$F(s,t)=\begin{cases}2s(1-t,t) & \text{if } s\in[0,1/2],\\ (1-t,t)+(2s-1)(t,1-t) & \text{if } s\in(1/2,1].\end{cases}$$

Then $F\circ G$ is a path homotopy from $f\cdot g$ to $h\cdot k$. □

Theorem 141. *[Problem 7-5] Let G be a topological group.*

(1) *Up to isomorphism, $\pi_1(G, g)$ is independent of the choice of the base point $g \in G$.*
(2) *$\pi_1(G, g)$ is abelian.*

Proof. Let $g_1, g_2 \in G$. The map $x \mapsto g_2 g_1^{-1} x$ is a homeomorphism of G with itself, so the induced map from $\pi_1(X, g_1) \to \pi_1(X, g_2)$ is an isomorphism. Therefore we can assume that $g = 1$ for part (2). Let f and g be loops based at $1 \in G$ and define $F : I \times I \to G$ by $(s, t) \mapsto f(s)g(t)$. Then $f \cdot g \sim g \cdot f$ by Theorem 140, so $[f] \cdot [g] = [g] \cdot [f]$. This shows that $\pi_1(G, 1)$ is abelian. □

Lemma 142. *Let $f_1, \ldots, f_n$ be paths in a topological space X such that $f_k(1) = f_{k+1}(0)$ for every $k = 1, \ldots, n-1$, and $f_n(1) = f_1(0)$. Let $f = f_1 \cdot \cdots \cdot f_n$ and $f' = f_n \cdot f_1 \cdot \cdots \cdot f_{n-1}$. Let $\widetilde{f}$ and $\widetilde{f'}$ be the circle representatives of f and f' respectively. Then $\widetilde{f}$ is (freely) homotopic to $\widetilde{f'}$.*

Proof. Let $\mu_k : I \to \mathbb{S}^1$ be given by $s \mapsto \exp((k + s)2\pi i/n)$. By reparametrizing $\widetilde{f}$ and $\widetilde{f'}$, we may assume that $f_k = \widetilde{f} \circ \mu_{k-1} = \widetilde{f'} \circ \mu_k$ for each $k = 1, \ldots, n$. Define $H : \mathbb{S}^1 \times I \to X$ by $H(z, t) = \widetilde{f}(e^{-2\pi i t/n} z)$ where z is taken to be a complex number. Then H is a homotopy from $\widetilde{f}$ to $\widetilde{f'}$. □

Theorem 143. *[Problem 7-6] For any path-connected space X and any base point $p \in X$, the map sending a loop to its circle representative induces a bijection between the set of conjugacy classes of elements of $\pi_1(X, p)$ and $[\mathbb{S}^1, X]$ (the set of free homotopy classes of continuous maps from $\mathbb{S}^1$ to X).*

Proof. Let C be the set of conjugacy classes of elements of $\pi_1(X, p)$ and denote the conjugacy class of an element $[f] \in \pi_1(X, p)$ by $[[f]]$. Let $\varphi : C \to [\mathbb{S}^1, X]$ be the map that sends an element $[[f]] \in C$ to the free homotopy class of the circle representative $\widetilde{f}$ of f. We first check that φ is well-defined. Let $[g] \in [[f]]$ so that $[g] = [\overline{h}] \cdot [f] \cdot [h]$ for some $[h] \in \pi_1(X, p)$. Then Lemma 142 shows that the circle representative of $\overline{h} \cdot f \cdot h$ is (freely) homotopic to the circle representative of $h \cdot \overline{h} \cdot f \sim f$, so $\widetilde{g} \simeq \widetilde{f}$. Let $\alpha : \mathbb{S}^1 \to X$ be a continuous map, let $\omega : I \to X$ be given by $\omega(s) = \alpha(e^{2\pi i s})$, and let γ be a path from p to $\omega(0)$. We have $\omega \simeq \overline{\gamma} \cdot \gamma \cdot \omega$ and by Lemma 142 the circle representative of $\overline{\gamma} \cdot \gamma \cdot \omega$ is homotopic to the circle representative of $\gamma \cdot \omega \cdot \overline{\gamma}$, so $\alpha \in \varphi([[\gamma \cdot \omega \cdot \overline{\gamma}]])$. This shows that φ is surjective. Finally, suppose that $\varphi([[f]]) = \varphi([[g]])$ so that $\widetilde{f}$ is homotopic to $\widetilde{g}$. Then $[f] = [g]$, and in particular we have $[[f]] = [[g]]$. This shows that φ is bijective. □

Theorem 144. *[Problem 7-7] Suppose (M_1, d_1) and (M_2, d_2) are metric spaces. A map $f : M_1 \to M_2$ is said to be* ***uniformly continuous*** *if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in M_1$, $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. If M_1 is compact, then every continuous map $f : M_1 \to M_2$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. For each $t \in M_1$, choose a number $\delta(t)$ such that $d_2(f(x), f(t)) < \varepsilon/2$ whenever $x \in M_1$ and $d_1(x,t) < \delta(t)$. Since M_1 is compact, the open cover $\mathcal{U} = \left\{ B_{\delta(t)}(t) : t \in M_1 \right\}$ of M_1 has a Lebesgue number δ. If $x \in M_1$ then the set $B_\delta(x)$ is contained in some $B_{\delta(t)}(t) \in \mathcal{U}$, so for all y with $d_1(x,y) < \delta$ we have

$$d_2(f(x), f(y)) \leq d_2(f(x), f(t)) + d_2(f(t), f(y)) < \varepsilon.$$

□

Theorem 145. *[Problem 7-8] A retract of a Hausdorff space is a closed subset.*

Proof. Let X be a Hausdorff space and let $r : X \to A$ be a retraction. Let $x \in X \setminus A$. Since X is Hausdorff, there exist neighborhoods U of x and V of $r(x)$ such that $U \cap V = \emptyset$. Now $U \cap r^{-1}(V \cap A)$ is a neighborhood of x contained in $X \setminus A$, since $r(A \cap U \cap r^{-1}(V \cap A)) \subseteq A \cap U \cap V \cap A = \emptyset$. □

Theorem 146. *[Problem 7-9] Suppose X and Y are connected topological spaces, and the fundamental group of Y is abelian. If $F, G : X \to Y$ are homotopic maps such that $F(x) = G(x)$ for some $x \in X$, then $F_* = G_* : \pi_1(X,x) \to \pi_1(Y, F(x))$.*

Proof. Let $H : X \times I \to Y$ be a homotopy from F to G and let $\alpha : I \to X$ be a loop based at x. Consider the map $H \circ (\alpha \times \mathrm{Id}_I) : I \times I \to Y$. By Lemma 7.17 we have $f \cdot g \sim h \cdot k$ where $f = F \circ \alpha$, $g(s) = h(s) = H(x,s)$, and $k(s) = G \circ \alpha$. But $[f] \cdot [g] = [g] \cdot [f] = [g] \cdot [k]$ implies that $[F \circ \alpha] = [f] = [k] = [G \circ \alpha]$ since $\pi_1(Y, F(x))$ is abelian, which shows that $F_* = G_*$. □

Theorem 147. *Let X and Y be topological spaces. If $F : X \to Y$ is null-homotopic then $F_* : \pi_1(X,x) \to \pi_1(Y, F(x))$ is the trivial map for all $x \in X$.*

Proof. Let $H : X \times I \to Y$ be a homotopy from F to a constant map and let $\alpha : I \to X$ be a loop based at x. Consider the map $H \circ (\alpha \times \mathrm{Id}_I) : I \times I \to Y$. By Lemma 7.17 we have $f \cdot g \sim h \cdot k$ where $f = F \circ \alpha$, $g(s) = h(s) = H(x,s)$, and k is a constant path. We have $f \sim f \cdot g \cdot \overline{g} \sim g \cdot k \cdot \overline{g} \sim c_{F(x)}$ from Theorem 7.11, and therefore F_* is trivial. □

Theorem 148. *[Problem 7-10] Let X and Y be topological spaces. If either X or Y is contractible, then every continuous map from X to Y is homotopic to a constant map.*

Proof. Let $f : X \to Y$ be a continuous map. If Id_X is homotopic to a constant map c then $f = f \circ \mathrm{Id}_X \simeq f \circ c$, and $f \circ c$ is a constant map. Similarly, if Id_Y is homotopic to a constant map c then $f = \mathrm{Id}_Y \circ f \simeq c \circ f$, and $c \circ f$ is a constant map. □

Theorem 149. *[Problem 7-11] The Möbius band is homotopy equivalent to $\mathbb{S}^1$.*

Proof. We define the Möbius band B to be the geometric realization of the presentation $\langle a, b, c \mid abcb \rangle$. In other words, it is the quotient space formed from $I \times I$ by identifying the edge $\{0\} \times I$ with $\{1\} \times I$. Let $q : I \times I \to B$ be the associated quotient map and define $H : (I \times I) \times I \to B$ by

$$H((x, y), t) = q\left(x, \frac{1}{2} + t\left(y - \frac{1}{2}\right)\right).$$

Then H descends to a continuous map $\widetilde{H} : B \times I \to B$ and in fact $\widetilde{H}$ is a (strong) deformation retraction from B to $q(I \times \{1/2\}) \approx \mathbb{S}^1$. □

Example 150. [Problem 7-12] Let X be the space of Example 5.9.

(1) $\{(0, 0)\}$ is a strong deformation retract of X.
(2) $\{(1, 0)\}$ is a deformation retract of X, but not a strong deformation retract.

The map $H : X \times I \to X$ given by

$$H(x, t) = \begin{cases} x/(1-t) & \text{if } 0 \le t < 1, \\ (0, 0) & \text{if } t = 1 \end{cases}$$

is a strong deformation retraction from X to $\{(0, 0)\}$. Furthermore,

$$H'(x, t) = \begin{cases} H(x, 2t) & \text{if } 0 \le t \le 1/2, \\ (2t - 1, 0) & \text{if } 1/2 < t \le 1. \end{cases}$$

is a deformation retraction from X to $\{(1, 0)\}$. However, $\{(1, 0)\}$ cannot be a strong deformation retract of X since it is a limit point of X but no retraction is possible directly towards $(1, 0)$.

Theorem 151. *[Problem 7-14] Let M be a compact connected surface that is not homeomorphic to $\mathbb{S}^2$. Then there is a point $p \in M$ such that $M \setminus \{p\}$ is homotopy equivalent to a bouquet of circles.*

Proof. This follows from Theorem 6.15. □

Theorem 152. *[Problem 7-16] Given any family $(X_\alpha)_{\alpha \in A}$ of topological spaces, the disjoint union space $\coprod_\alpha X_\alpha$ is their coproduct in the category* Top.

Proof. This follows from Theorem 59. □

Theorem 153. *[Problem 7-17] The wedge sum is the coproduct in the category* Top_*.

Proof. Let $((X_\alpha, p_\alpha))_{\alpha \in A}$ be a family of pointed spaces, let $X = \bigvee_{\alpha \in A} X_\alpha$ and define $j_\alpha : X_\alpha \to X$ by $j_\alpha = q \circ i_\alpha$ where $i_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ and $q : \coprod_{\alpha \in A} X_\alpha \to \bigvee_{\alpha \in A} X_\alpha$ is the quotient map. Note that $j_\alpha(p_\alpha) = j_\beta(p_\beta)$ for all α, β; denote this common value by p. Let (W, r) be a pointed space and let $f_\alpha : X_\alpha \to W$ be a pointed continuous map.

There exists a unique continuous map $f : \coprod_{\alpha \in A} X_\alpha \to W$ such that $f_\alpha = f \circ i_\alpha$ for all α. Since $f(p_\alpha) = r$ for all α, there exists a unique continuous map $g : \bigvee_{\alpha \in A} X_\alpha \to W$ such that $f = g \circ q$. Then $f_\alpha = f \circ i_\alpha = g \circ q \circ i_\alpha = g \circ j_\alpha$ for all α. This shows that (X, p) is the coproduct of $((X_\alpha, p_\alpha))_{\alpha \in A}$. ☐

Theorem 154. *[Problem 7-18] Let $(G_\alpha)_{\alpha \in A}$ be a family of abelian groups. The direct sum, together with the obvious injections $i_\alpha : G_\alpha \hookrightarrow \bigoplus_\alpha G_\alpha$, is the coproduct of the G_α's in the category* Ab*.*

Proof. Let X be an abelian group and let $f_\alpha : G_\alpha \to X$ be homomorphisms. Let $f : \bigoplus_\alpha G_\alpha \to X$ be given by $(g_\alpha) \mapsto \sum_{\alpha \in A} f_\alpha(g_\alpha)$, which is well-defined since $g_\alpha = 0$ for all but finitely many α. It is clear that $f_\alpha = f \circ i_\alpha$ for all α. Suppose that f' is another homomorphism such that $f_\alpha = f' \circ i_\alpha$ for all α. Then for all $(g_\alpha) \in \bigoplus_\alpha G_\alpha$ we have

$$f'((g_\alpha)) = f'\left(\sum_{\alpha \in A} i_\alpha(g_\alpha)\right) = \sum_{\alpha \in A} (f' \circ i_\alpha)(g_\alpha) = \sum_{\alpha \in A} f_\alpha(g_\alpha),$$

which shows that $f' = f$. ☐

Remark 155. [Problem 7-19] The direct sum does not yield the coproduct in the category Grp. Take $G_1 = G_2 = \mathbb{Z}$, $H = \mathrm{GL}(2, \mathbb{R})$ and the maps $f_k : G_k \to H$ defined by

$$f_1(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad f_2(n) = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}.$$

There is no homomorphism $f : \mathbb{Z} \oplus \mathbb{Z} \to H$ such that $f_k = f \circ i_k$ for $k = 1, 2$ since

$$f((1,1)) = f((1,0) + (0,1)) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

but

$$f((1,1)) = f((0,1) + (1,0)) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Chapter 8. The Circle

Theorem 156. *[Exercise 8.7] A **rotation of** $\mathbb{S}^1$ is a map $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ of the form $\rho(z) = e^{i\theta} z$ for some fixed $e^{i\theta} \in \mathbb{S}^1$. If ρ is a rotation, then $N(\rho \circ f) = N(f)$ for every loop f in $\mathbb{S}^1$.*

Proof. Define $\widetilde{g} : \mathbb{S}^1 \to \mathbb{R}$ by $z \mapsto \theta/2\pi + \widetilde{f}(z)$. Then $\widetilde{g}$ is a lift of $\rho \circ f$, so

$$\begin{aligned} N(\rho \circ f) &= \widetilde{g}(1) - \widetilde{g}(0) \\ &= \theta/2\pi + \widetilde{f}(1) - \theta/2\pi - \widetilde{f}(0) \\ &= \widetilde{f}(1) - \widetilde{f}(0) \end{aligned}$$

$$= N(f).$$

□

Theorem 157. *[Problem 8-1] (cf. Theorem 80)*

(1) *If $U \subseteq \mathbb{R}^2$ is an open subset and $x \in U$, then $U \setminus \{x\}$ is not simply connected.*
(2) *If $n > 2$ then $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}^2$.*

Proof. Let $B \subseteq U$ be an open ball of radius $r > 0$ around x. A loop γ that traverses the circle ∂B counterclockwise once has a winding number of 1. If γ is null-homotopic in $U \setminus \{x\}$ then it is also null-homotopic in $\mathbb{R}^2 \setminus \{x\}$, which contradicts Corollary 8.11. This shows that $U \setminus \{x\}$ is not simply connected. Part (2) follows immediately from Corollary 7.38. □

Theorem 158. *[Problem 8-2] A nonempty topological space cannot be both a 2-manifold and an n-manifold for any $n > 2$ (cf. Theorem 81).*

Proof. Let M be a nonempty topological space that is both a 2-manifold and an n-manifold for some $n > 2$. Choose some $p \in M$ and let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homeomorphisms where U_1 and U_2 are neighborhoods of p, V_1 is open in $\mathbb{R}^2$, and V_2 is open in $\mathbb{R}^n$. Let B be an open ball around $\varphi_2(p)$ contained in $\varphi_2(U_1 \cap U_2)$. Then $W_1 = B \setminus \{\varphi_2(p)\}$ is homeomorphic to $W_2 = (\varphi_1 \circ \varphi_2^{-1})(B) \setminus \{\varphi_1(p)\}$, but W_2 is not simply connected by Theorem 157 while W_1 is simply connected. This is a contradiction. □

Theorem 159. *[Problem 8-3] Suppose M is a 2-dimensional manifold with boundary. Then the interior and boundary of M are disjoint (cf. Theorem 82).*

Proof. Suppose $p \in M$ is both an interior and boundary point. Choose coordinate charts (U, φ) and (V, ψ) such that U, V are neighborhoods of p, $\varphi(U)$ is open in $\operatorname{Int} \mathbb{H}^2$, $\psi(V)$ is open in $\mathbb{H}^2$, and $\psi(p) \in \partial \mathbb{H}^2$. Let $W = U \cap V$; then $\varphi(W)$ is homeomorphic to $\psi(W)$. But this is impossible, for $\varphi(W) \setminus \{\varphi(p)\}$ is not simply connected by Theorem 157 while $\psi(W) \setminus \{\psi(p)\}$ is simply connected. □

Theorem 160. *[Problem 8-4] A continuous map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ has an extension to a continuous map $\phi : \overline{\mathbb{B}^2} \to \mathbb{S}^1$ if and only if it has degree zero.*

Proof. Let $\omega : I \to \mathbb{S}^1$ be standard generator of $\pi_1(\mathbb{S}^1, 1)$. By Proposition 7.16, $\varphi \circ \omega$ has a winding number of zero if and only if φ extends to a continuous map from $\overline{\mathbb{B}^2}$ to $\mathbb{S}^1$. □

Theorem 161. *[Problem 8-5] Every nonconstant polynomial in one complex variable has a zero.*

Proof. Suppose that $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with no zeros and $n > 0$. We can assume that $a_0 \neq 0$, for otherwise $p(0) = 0$. Let

$$p_\varepsilon(z) = z^n + a_{n-1}\varepsilon z^{n-1} + \cdots + a_1 \varepsilon^{n-1} z + a_0 \varepsilon^n$$

so that $p_\varepsilon(z) = \varepsilon^n p(z/\varepsilon)$ when $\varepsilon \neq 0$. Since p has no zeros, the map $H : \mathbb{S}^1 \times I \to \mathbb{C} \setminus \{0\}$ given by $H(z,t) = p_{t\varepsilon}(z)$ is a homotopy from $z \mapsto z^n$ to $p_\varepsilon|_{\mathbb{S}^1}$. Therefore $p_\varepsilon|_{\mathbb{S}^1}$ has a winding number of n, and the map $\phi : \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$ given by $\phi(z) = p(z/\varepsilon) = \varepsilon^{-n} p_\varepsilon(z)$ also has a winding number of n. But ϕ is homotopic to the constant loop $c_{p(0)} = c_{a_0}$ by the homotopy $(z,t) \mapsto p(tz/\varepsilon)$, which is a contradiction. □

Theorem 162. *[Problem 8-6] Every continuous map $f : \overline{\mathbb{B}^2} \to \overline{\mathbb{B}^2}$ has a fixed point.*

Proof. If f has no fixed point then we can define a continuous map $\varphi : \overline{\mathbb{B}^2} \to \mathbb{S}^1$ by

$$\varphi(x) = \frac{x - f(x)}{\|x - f(x)\|};$$

by Theorem 160, $\varphi|_{\mathbb{S}^1}$ has degree zero. Define $H : \mathbb{S}^1 \times I \to \mathbb{S}^1$ by

$$H(x,t) = \frac{x - (1-t)f(x)}{\|x - (1-t)f(x)\|}.$$

If $t = 0$ then the denominator is never zero. Otherwise,

$$\|x - (1-t)f(x)\| \geq \|\|x\| - (1-t)\|f(x)\|\| \geq t > 0$$

for $t \in (0,1]$ since $\|x\| = 1$ and $\|f(x)\| \leq 1$. This shows that H is a well-defined homotopy from $\varphi|_{\mathbb{S}^1}$ to $\mathrm{Id}_{\mathbb{S}^1}$, which contradicts the fact that $\varphi|_{\mathbb{S}^1}$ has degree zero. □

Lemma 163. *Let $f : I \to \mathbb{S}^1$ be a loop with winding number n. Then there exists a lift $\widetilde{f} : I \to \mathbb{R}$ of f such that $\widetilde{f}(0) = 0$ and $\widetilde{f}(1) = n$.*

Proof. We can assume that f is based at $1 \in \mathbb{S}^1$. Let $\alpha : I \to \mathbb{S}^1$ be the map $s \mapsto e^{2\pi i n s}$ and let $\widetilde{\alpha} : I \to \mathbb{R}$ be the lift of α given by $s \mapsto ns$. By Corollary 8.5, there exists a lift $\widetilde{f}$ of f such that $\widetilde{f}(0) = 0 = \widetilde{\alpha}(0)$, and since $\alpha \sim f$, Corollary 8.6 shows that $\widetilde{f}(1) = \widetilde{\alpha}(1) = n$. □

Theorem 164. *[Problem 8-7] If $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous and* $\deg \varphi \neq \pm 1$*, then φ is not injective.*

Proof. The result is clear when $\deg \varphi = 0$, so assume otherwise; since $\deg \varphi \neq \pm 1$, we have $|\deg \varphi| > 1$. Let $\varepsilon : I \to \mathbb{S}^1$ be the map $s \mapsto e^{2\pi i s}$. By Lemma 163, there exists a lift $\widetilde{\varphi} : I \to \mathbb{R}$ of $\varphi \circ \varepsilon$ such that $\widetilde{\varphi}(0) = 0$ and $|\widetilde{\varphi}(1)| = |\deg \varphi| > 1$. Choose a point $0 < s < 1$ such that $\widetilde{\varphi}(s) = \pm 1$. Noting that $\varepsilon \circ \widetilde{\varphi} = \varphi \circ \varepsilon$, we have $\varphi(\varepsilon(0)) = \varphi(\varepsilon(s))$ while $\varepsilon(0) \neq \varepsilon(s)$, which shows that φ is not injective. □

Theorem 165. *[Problem 8-8] Suppose $\varphi, \psi : \mathbb{S}^1 \to \mathbb{S}^1$ are continuous maps of different degrees. Then there is a point $z \in \mathbb{S}^1$ where $\varphi(z) = -\psi(z)$.*

Proof. This follows from Theorem 137. □

Theorem 166. *[Problem 8-9] Suppose $f : I \to \mathbb{C} \setminus \{0\}$ is a continuously differentiable loop. Then its winding number is given by*

$$N(f) = \frac{1}{2\pi i} \int_0^1 \frac{f'(s)}{f(s)}\, ds.$$

Proof. Let $\widetilde{f} : I \to \mathbb{R}$ be a lift of $f/|f|$; then $\exp(2\pi i \widetilde{f}(s)) = f(s)/r(s)$ for all $s \in I$, where $r = |f|$. Now

$$\begin{aligned}
\frac{1}{2\pi i} \int_0^1 \frac{f'(s)}{f(s)}\, ds &= \frac{1}{2\pi i} \int_0^1 \frac{2\pi i r(s) \widetilde{f}'(s) \exp(2\pi i \widetilde{f}(s)) + r'(s) \exp(2\pi i \widetilde{f}(s))}{r(s) \exp(2\pi i \widetilde{f}(s))}\, ds \\
&= \frac{1}{2\pi i} \int_0^1 \left(2\pi i \widetilde{f}'(s) + \frac{r'(s)}{r(s)} \right) ds \\
&= \frac{1}{2\pi i} \left[2\pi i \widetilde{f}(s) + \log r(s) \right]_0^1 \\
&= \frac{1}{2\pi i} [2\pi i \widetilde{f}(1) - 2\pi i \widetilde{f}(0) + \log r(1) - \log r(0)] \\
&= \widetilde{f}(1) - \widetilde{f}(0) \\
&= N(f).
\end{aligned}$$

□

Theorem 167. *[Problem 8-10] A **vector field** on $\mathbb{R}^n$ is a continuous map $V : \mathbb{R}^n \to \mathbb{R}^n$. If V is a vector field, a point $p \in \mathbb{R}^n$ is called a **singular point of** V if $V(p) = 0$, and a **regular point** if $V(p) \neq 0$. A singular point is **isolated** if it has a neighborhood containing no other singular points. Suppose V is a vector field on $\mathbb{R}^2$, and let $\mathcal{R}_V \subseteq \mathbb{R}^2$ denote the set of regular points of V. For any loop $f : I \to \mathcal{R}_V$, define the **winding number of** V **around** f, denoted by $N(V, f)$, to be the winding number of the loop $V \circ f : I \to \mathbb{R}^2 \setminus \{0\}$.*

1. *$N(V, f)$ depends only on the path class of f.*
2. *Suppose p is an isolated singular point of V. Then $N(V, f_\varepsilon)$ is independent of ε for ε sufficiently small, where $f_\varepsilon(s) = p + \varepsilon\omega(s)$, and ω is the standard counterclockwise loop around the unit circle. This integer is called the **index of** V **at** p, and is denoted by $\operatorname{Ind}(V, p)$.*
3. *Now assume V has finitely many singular points in the open unit disk. Then the index of V around the loop ω is equal to the sum of the indices of V at the interior singular points.*

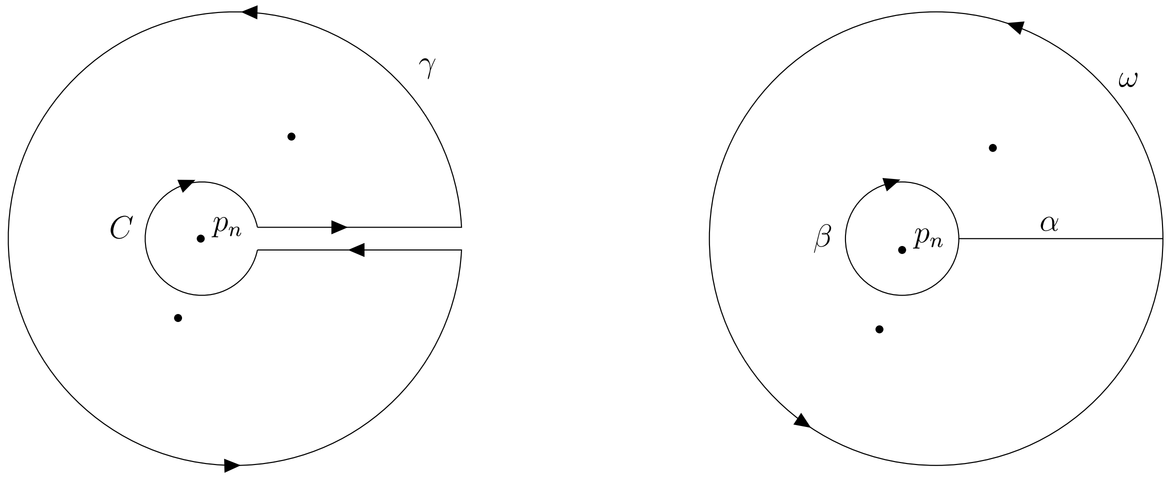


FIGURE 0.1. A keyhole path.

Proof. Part (1) follows from Proposition 7.2. For (2), choose $r > 0$ so that $B_r(p)$ has no other singular points. If $0 < \varepsilon_1, \varepsilon_2 < r$ then $H : I \times I \to \mathcal{R}_V$ given by $H(s,t) = p+[(1-t)\varepsilon_1+t\varepsilon_2]\omega(s)$ is a homotopy from f_{ε_1} to f_{ε_2}, and $N(V, f_{\varepsilon_1}) = N(V, f_{\varepsilon_2})$ by part (1). For part (3), we use induction on the number n of singular points. The result is clear when $n = 1$. Assuming the result for $n-1$, suppose we have n singular points $p_1, \ldots, p_n$. Let γ be a keyhole path in $\mathbb{B}^2 \backslash \{p_1, \ldots, p_n\}$ that traverses the unit disk counterclockwise and traverses a small circle C around p_n clockwise, such that γ encloses $p_1, \ldots, p_{n-1}$ (see Figure 0.1). Then $N(V, \gamma)$ encloses $p_1, \ldots, p_{n-1}$, and ignoring p_n, γ is homotopic to ω. Therefore $N(V, \gamma) = \sum_{i=1}^{n-1} \operatorname{Ind}(V, p_i)$ by the induction hypothesis. Let $\widehat{\gamma}$ be the path γ with the gap closed, so that $\widehat{\gamma}$ is homotopic to both γ and $\omega \cdot \alpha \cdot \beta \cdot \overline{\alpha}$ where α is a path from $1 \in \mathbb{S}^1$ to a point x on C and β is a clockwise loop around C based at x. By applying Theorem 166 we have $N(V, \widehat{\gamma}) = N(V, \omega \cdot \alpha \cdot \beta \cdot \overline{\alpha}) = N(V, \omega) + N(V, \beta)$ since α cancels $\overline{\alpha}$ in the integral. But $N(V, \beta) = -\operatorname{Ind}(V, p_n)$, so

$$\begin{aligned} N(V, \omega) &= N(V, \widehat{\gamma}) - N(V, \beta) \\ &= N(V, \gamma) - N(V, \beta) \\ &= \sum_{i=1}^{n} \operatorname{Ind}(V, p_i). \end{aligned}$$

□

Theorem 168. *[Problem 8-11] For each continuous map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ there is a 2×2 integer matrix $D(\varphi)$ with the following properties:*

(1) *$D(\psi \circ \varphi)$ is equal to the matrix product $D(\psi)D(\varphi)$.*
(2) *Two continuous maps φ and ψ are homotopic if and only if $D(\varphi) = D(\psi)$.*

(3) *For every 2×2 integer matrix E, there is a continuous map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $D(\varphi) = E$. If E is invertible then φ is a homeomorphism.*
(4) *φ is homotopic to a homeomorphism only if and only if $D(\varphi)$ is invertible over the integers.*

Proof. We can consider $\pi_1(\mathbb{T}^2, (1,1))$ as a free $\mathbb{Z}$-module with basis $\{[\omega_1], [\omega_2]\}$ where ω_j is the standard loop in the jth copy of $\mathbb{S}^1$. For a continuous map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\varphi(1,1) = (1,1)$, define $D(\varphi)$ to be the unique 2×2 integer matrix that represents the $\mathbb{Z}$-map

$$\varphi_* : \pi_1\left(\mathbb{T}^2, (1,1)\right) \to \pi_1\left(\mathbb{T}^2, (1,1)\right).$$

Note that if $i_j : \mathbb{S}^1 \to \mathbb{T}^2$ and $p_j : \mathbb{T}^2 \to \mathbb{S}^1$ are the canonical injections and projections then

$$(*) \qquad D(\varphi) = \begin{bmatrix} \nu_{11}(1) & \nu_{12}(1) \\ \nu_{21}(1) & \nu_{22}(1) \end{bmatrix}$$

where $\varphi_{jk} = p_j \circ \varphi \circ i_k$ and $\nu_{jk} : \mathbb{Z} \to \mathbb{Z}$ is the unique homomorphism such that $(\varphi_{jk})_*([\omega]) = [\omega]^{\nu_{jk}(1)}$. In other words, the entries of the matrix $D(\varphi)$ are the degrees of the component maps φ_{jk}. Part (1) follows immediately since $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. If φ and ψ are homotopic then $\varphi_{jk} \simeq \psi_{jk}$ for $1 \le j,k \le 2$, and $D(\varphi) = D(\psi)$ from (*). Conversely, if $D(\varphi) = D(\psi)$ then $\varphi_{jk} \simeq \psi_{jk}$ for $1 \le j,k \le 2$, so $\varphi \simeq \psi$ by Theorem 136.

For part (3), let E_{jk} be the entries of E. If we set $\varphi_{jk}(s) = s^{E_{jk}}$, then there exist unique continuous maps $\varphi_k : \mathbb{S}^1 \to \mathbb{T}^2$ such that $\varphi_{jk} = p_j \circ \varphi_k$ for each j. Define $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ by $\varphi(s_1, s_2) = \varphi_1(s_1)\varphi_2(s_2)$ where the multiplication is the component-wise multiplication of complex numbers; then $D(\varphi) = E$. More explicitly, we have

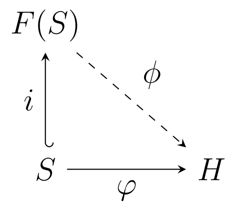
$$\varphi \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \varphi_{11}(s_1)\varphi_{12}(s_2) \\ \varphi_{21}(s_1)\varphi_{22}(s_2) \end{bmatrix} = \begin{bmatrix} s_1^{E_{11}} s_2^{E_{12}} \\ s_1^{E_{21}} s_2^{E_{22}} \end{bmatrix}.$$

If $\varphi_E : \mathbb{T}^2 \to \mathbb{T}^2$ denotes the above map for the matrix E then $\varphi_{EF} = \varphi_E \circ \varphi_F$ by direct computation. It follows that φ is a homeomorphism if E is invertible.

Finally, if φ is homotopic to a homeomorphism ψ then $\psi^{-1} \circ \varphi \simeq \psi^{-1} \circ \psi = \mathrm{Id}_{\mathbb{T}^2}$, so $D(\psi^{-1} \circ \varphi) = D(\psi^{-1})D(\varphi) = I$ and $D(\varphi)$ is invertible. Conversely, if $D(\varphi)$ is invertible then there is a homeomorphism ψ such that $D(\psi) = D(\varphi)$ by part (3), so $\varphi \simeq \psi$.

For the general case when $\varphi(1,1)$ is not necessarily equal to $(1,1)$, define $D(\varphi) = D(\rho_\varphi \circ \varphi)$ where ρ_φ is the rotation $(z_1, z_2) \mapsto \varphi(1,1)^{-1}(z_1, z_2)$ that takes $\varphi(1,1)$ to $(1,1)$; the preceding proofs can then be easily modified. $\square$

Theorem 169. *[Exercise 9.10] Let S be a set. For any group H and any map $\varphi : S \to H$, there exists a unique homomorphism $\phi : F(S) \to H$ extending φ:*



Proof. For each $\sigma \in S$ there is a unique homomorphism $\varphi_\sigma : F(\sigma) \to H$ such that $\varphi_\sigma(\sigma) = \varphi(\sigma)$. If $i_\sigma : F(\sigma) \to F(S)$ is the canonical injection, by Theorem 9.5 there is a unique homomorphism $\phi : F(S) \to H$ such that $\varphi_\sigma = \phi \circ i_\sigma$ for every $\sigma \in S$. Since $\phi(\sigma) = (\phi \circ i_\sigma)(\sigma) = \varphi(\sigma)$, the map ϕ extends φ. Furthermore, if $\phi' : F(S) \to H$ is another homomorphism that extends φ then

$$\varphi_\sigma(\sigma^n) = \varphi(\sigma)^n = [(\phi' \circ i_\sigma)(\sigma)]^n = (\phi' \circ i_\sigma)(\sigma^n)$$

for every $\sigma \in S$, so $\phi' = \phi$. □

Theorem 170. *[Exercise 9.11] The free group on S is the unique group (up to isomorphism) satisfying the characteristic property of Theorem 169.*

Proof. Let G and G' be two groups satisfying the characteristic property, and let $i : S \to G$ and $i' : S \to G'$ be the inclusion maps. There exist unique homomorphisms $\varphi : G' \to G$ and $\varphi' : G \to G'$ such that $i = \varphi \circ i'$ and $i' = \varphi' \circ i$. Then $\varphi \circ \varphi' \circ i = \varphi \circ i' = i$, so $\varphi \circ \varphi' = \mathrm{Id}_G$ since both $\varphi \circ \varphi'$ and Id_G satisfy the diagram with G in place of $F(S)$ and H. Similarly, $\varphi' \circ \varphi = \mathrm{Id}_{G'}$. This shows that $G \cong G'$. □

Theorem 171. *[Exercise 9.15] Let S be a nonempty set.*

1. *Given any abelian group H and any map $\varphi : S \to H$, there exists a unique homomorphism $\phi : \mathbb{Z}S \to H$ extending φ.*
2. *The free abelian group $\mathbb{Z}\{\sigma_1, \ldots, \sigma_n\}$ on a finite set is isomorphic to $\mathbb{Z}^n$ via the map $(k_1, \ldots, k_n) \mapsto k_1\sigma_1 + \cdots + k_n\sigma_n$.*

Proof. If $\phi : \mathbb{Z}S \to H$ is a homomorphism extending φ then

$$\begin{aligned}\phi(k_1\sigma_1 + \cdots + k_n\sigma_n) &= k_1\phi(\sigma_1) + \cdots + k_n\phi(\sigma_n)\\ &= k_1\varphi(\sigma_1) + \cdots + k_n\varphi(\sigma_n)\end{aligned}$$

for all $k_1, \ldots, k_n \in \mathbb{Z}$ and $\sigma_1, \ldots, \sigma_n \in S$, which shows that ϕ is unique. But the above equation clearly defines a homomorphism, so ϕ exists. Part (2) is obvious. □

Theorem 172. *[Exercise 9.16] For any set S, the identity map of S induces an isomorphism between the free abelian group on S and the direct sum of infinite cyclic groups generated by elements of S:*

$$\mathbb{Z}S \cong \bigoplus_{\sigma \in S} \mathbb{Z}\{\sigma\}.$$

Proof. This is clear from the definition of the direct sum. □

Theorem 173. *[Exercise 9.17]*

(1) *An abelian group is free abelian if and only if it has a basis.*
(2) *Any two free abelian groups whose bases have the same cardinality are isomorphic.*

Proof. If G is a free abelian group then there is a subset $S \subseteq G$ such that the inclusion $S \hookrightarrow G$ induces an isomorphism $\varphi : \mathbb{Z}S \to G$. Then $\{\varphi(\sigma) : \sigma \in S\}$ is clearly a basis for G. Conversely, if an abelian group G has a basis S then the induced map $\varphi : \mathbb{Z}S \to G$ has trivial kernel and is surjective. For (2), let G and G' be two free abelian groups with bases B and B' respectively; assume that there is a bijection $\varphi : B \to B'$. Since $G \cong \mathbb{Z}B$ and $G' \cong \mathbb{Z}B'$, it suffices to show that $\mathbb{Z}B \cong \mathbb{Z}B'$. If $i : B \to \mathbb{Z}B$ and $i' : B' \to \mathbb{Z}B'$ are the canonical injections, there exist unique homomorphisms $\phi : \mathbb{Z}B' \to \mathbb{Z}B$ and $\phi' : \mathbb{Z}B \to \mathbb{Z}B'$ such that $i' \circ \varphi = \phi' \circ i$ and $i \circ \varphi^{-1} = \phi \circ i'$. Then $\phi \circ \phi' \circ i = \phi \circ i' \circ \varphi = i \circ \varphi^{-1} \circ \varphi = i$, so $\phi \circ \phi' = \mathrm{Id}_{\mathbb{Z}B}$ by uniqueness in Theorem 171. Similarly, $\phi' \circ \phi = \mathrm{Id}_{\mathbb{Z}B'}$. This shows that $\mathbb{Z}B \cong \mathbb{Z}B'$. □

Theorem 174. *[Problem 9-1] The free product of two or more nontrivial groups is infinite and nonabelian.*

Proof. Let G_1 and G_2 be nontrivial groups. Choose non-identity elements $g_1 \in G_1$ and $g_2 \in G_2$; then $w_n = \prod_{i=1}^{n} g_1 g_2$ produces an infinite sequence of distinct elements in $G_1 * G_2$. If $w_n' = g_1 \prod_{i=1}^{n} g_2^{-1} g_1^{-1}$ then $w_n' w_n = g_1$ but $w_n w_n' \neq g_1$. □

Theorem 175. *[Problem 9-2] A free group on two or more generators has center consisting of the identity alone.*

Proof. We can assume that there are exactly two generators x and y. Let $x^m w y^n \in F(x, y)$ where $m \neq 0$ or $n \neq 0$ and w is some word. Suppose $m \neq 0$. Then

$$(x^m w y^n)(y^{-n} w^{-1} x^{-m} y) = y$$

but

$$(y^{-n} w^{-1} x^{-m} y)(x^m w y^n) \neq y.$$

Similarly, if $n \neq 0$ then $x^m w y^n$ and $x y^{-n} w^{-1} x^{-m}$ do not commute. □

Theorem 176. *[Problem 9-3] A group G is free if and only if it has a generating set $S \subseteq G$ such that every element $g \in G$ other than the identity has a unique expression as a product of the form*

$$g = \sigma_1^{n_1} \cdots \sigma_k^{n_k}$$

where $\sigma_i \in S$, n_i are nonzero integers, and $\sigma_i \neq \sigma_{i+1}$ for each $i = 1, \dots, k-1$.

Proof. If G is free with basis $S \subseteq G$ then the result follows from Proposition 9.2, since any expression of the indicated form is a reduced word. Conversely, let $S \subseteq G$. If S generates G then the induced homomorphism $\phi : F(S) \to G$ is surjective; if S also satisfies the unique expression property then ϕ is injective. $\square$

Theorem 177. *[Problem 9-4] Let G_1, G_2, H_1, H_2 be groups, and let $f_j : G_j \to H_j$ be group homomorphisms for $j = 1, 2$.*

(1) *There exists a unique homomorphism $f_1 * f_2 : G_1 * G_2 \to H_1 * H_2$ such that the following diagram commutes for $j = 1, 2$:*

$$\begin{array}{ccc} G_1 * G_2 & \xrightarrow{f_1 * f_2} & H_1 * H_2 \\ {\scriptstyle i_j}\uparrow & & \uparrow{\scriptstyle i'_j} \\ G_j & \xrightarrow[f_j]{} & H_j \end{array}$$

where $i_j : G_j \to G_1 * G_2$ and $i'_j : H_j \to H_1 * H_2$ are the canonical injections.

(2) *Let S_1 and S_2 be disjoint sets, and let R_i be a subset of the free group $F(S_i)$ for $i = 1, 2$. Then $\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$ is a presentation of the free product group $\langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle$.*

Proof. Consider the homomorphisms $i'_j \circ f_j : G_j \to H_1 * H_2$ for $j = 1, 2$. By the characteristic property there exists a unique homomorphism $f_1 * f_2 : G_1 * G_2 \to H_1 * H_2$ such that $i'_j \circ f_j = (f_1 * f_2) \circ i_j$ for $j = 1, 2$, which proves (1).

For (2), we first show that $F(S_1 \cup S_2) \cong F(S_1) * F(S_2)$. For $i = 1, 2$, let

$$\begin{aligned} j_i &: S_i \to S_1 \cup S_2, \\ k_i &: S_i \to F(S_i), \\ \ell_i &: F(S_i) \to F(S_1) * F(S_2), \\ j &: S_1 \cup S_2 \to F(S_1 \cup S_2) \end{aligned}$$

be the canonical injections. There exist unique homomorphisms $m_i : F(S_i) \to F(S_1 \cup S_2)$ such that $m_i \circ k_i = j \circ j_i$, and these maps induce a unique homomorphism $\varphi : F(S_1) * F(S_2) \to F(S_1 \cup S_2)$ satisfying $m_i = \varphi \circ \ell_i$. Since S_1 and S_2 are disjoint, there is a unique map $k : S_1 \cup S_2 \to F(S_1) * F(S_2)$ satisfying $k \circ j_i = \ell_i \circ k_i$. This induces a unique homomorphism $\psi : F(S_1 \cup S_2) \to F(S_1) * F(S_2)$ satisfying $k = \psi \circ j$. Now

$$\varphi \circ \psi \circ j \circ j_i = \varphi \circ k \circ j_i = \varphi \circ \ell_i \circ k_i = m_i \circ k_i = j \circ j_i,$$

so $\varphi \circ \psi \circ j = j$ and $\varphi \circ \psi = \mathrm{Id}_{F(S_1 \cup S_2)}$ by uniqueness. Similarly,

$$\psi \circ \varphi \circ \ell_i \circ k_i = \psi \circ m_i \circ k_i = \psi \circ j \circ j_i = k \circ j_i = \ell_i \circ k_i,$$

so $\psi \circ \varphi \circ \ell_i = \ell_i$ and $\psi \circ \varphi = \mathrm{Id}_{F(S_1) * F(S_2)}$ by uniqueness. This proves that φ is an isomorphism. We now show that

$$F(S_1 \cup S_2)/\overline{R_1 \cup R_2} \cong (F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2}).$$

Let

$$\pi : F(S_1 \cup S_2) \to F(S_1 \cup S_2)/\overline{R_1 \cup R_2},$$
$$\pi_i : F(S_i) \to F(S_i)/\overline{R_i}$$

be the quotient maps and let

$$n_i : F(S_i)/\overline{R_i} \to (F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2})$$

be the canonical injections. Since $\overline{R_i} \subseteq \ker(\pi \circ m_i)$, there are unique homomorphisms $f_i : F(S_i)/\overline{R_i} \to F(S_1 \cup S_2)/\overline{R_1 \cup R_2}$ such that $f_i \circ \pi_i = \pi \circ m_i$. Then there is a unique homomorphism $f : (F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2}) \to F(S_1 \cup S_2)/\overline{R_1 \cup R_2}$ such that $f_i = f \circ n_i$. There is a unique homomorphism $p : F(S_1) * F(S_2) \to (F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2})$ satisfying $n_i \circ \pi_i = p \circ \ell_i$. Since $\overline{R_1 \cup R_2} \subseteq \ker(p \circ \psi)$, there is a unique homomorphism $g : F(S_1 \cup S_2)/\overline{R_1 \cup R_2} \to (F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2})$ satisfying $g \circ \pi = p \circ \psi$. Now

$$\begin{aligned} g \circ f \circ n_i \circ \pi_i &= g \circ f_i \circ \pi_i = g \circ \pi \circ m_i \\ &= p \circ \psi \circ m_i = p \circ \psi \circ \varphi \circ \ell_i = p \circ \ell_i = n_i \circ \pi_i, \end{aligned}$$

so $g \circ f \circ n_i = n_i$ and $g \circ f = \mathrm{Id}_{(F(S_1)/\overline{R_1}) * (F(S_2)/\overline{R_2})}$ by uniqueness. Similarly,

$$\begin{aligned} f \circ g \circ \pi \circ m_i &= f \circ p \circ \psi \circ m_i = f \circ p \circ \psi \circ \varphi \circ \ell_i \\ &= f \circ p \circ \ell_i = f \circ n_i \circ \pi_i = f_i \circ \pi_i = \pi \circ m_i \end{aligned}$$

implies that

$$f \circ g \circ \pi \circ j \circ j_i = f \circ g \circ \pi \circ m_i \circ k_i = \pi \circ m_i \circ k_i = \pi \circ j \circ j_i,$$

so $f \circ g \circ \pi = \pi$ and $f \circ g = \mathrm{Id}_{F(S_1 \cup S_2)/\overline{R_1 \cup R_2}}$ by uniqueness. This proves that

$$\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle \cong \langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle .$$

□

Theorem 178. *[Problem 9-5] Let S be a set, let R and R' be subsets of the free group $F(S)$, and let $\pi : F(S) \to \langle S \mid R \rangle$ be the projection onto the quotient group. Then $\langle S \mid R \cup R' \rangle$ is a presentation of the quotient group $\langle S \mid R \rangle / \overline{\pi(R')}$.*

Proof. We want to show that

$$F(S)/\overline{R \cup R'} \cong (F(S)/\overline{R})/\overline{\pi(R')}.$$

Let

$$\pi' : F(S) \to F(S)/\overline{R \cup R'},$$
$$\pi'' : F(S)/\overline{R} \to (F(S)/\overline{R})/\overline{\pi(R')}$$

be the quotient maps. Since $\overline{R} \subseteq \ker(\pi')$, there is a unique homomorphism $f_1 : F(S)/\overline{R} \to F(S)/\overline{R \cup R'}$ such that $\pi' = f_1 \circ \pi$. But $\overline{\pi(R')} \subseteq \ker(f_1)$ since $R' \subseteq \ker(\pi')$, so there is a unique homomorphism $f : (F(S)/\overline{R})/\overline{\pi(R')} \to F(S)/\overline{R \cup R'}$ such that $f_1 = f \circ \pi''$. Since $\overline{R \cup R'} \subseteq \ker(\pi'' \circ \pi)$, there exists a unique homomorphism $g : F(S)/\overline{R \cup R'} \to (F(S)/\overline{R})/\overline{\pi(R')}$ such that $\pi'' \circ \pi = g \circ \pi'$. Now

$$f \circ g \circ \pi' = f \circ \pi'' \circ \pi = f_1 \circ \pi = \pi'$$

and

$$g \circ f \circ \pi'' \circ \pi = g \circ f_1 \circ \pi = g \circ \pi' = \pi'' \circ \pi,$$

so $f \circ g = \mathrm{Id}_{F(S)/\overline{R \cup R'}}$ and $g \circ f = \mathrm{Id}_{(F(S)/\overline{R})/\overline{\pi(R')}}$ by uniqueness. □

Theorem 179. *[Problem 9-6]*

(1) *The free group on generators $\alpha_1, \ldots, \alpha_n$ has the presentation*

$$F(\alpha_1, \ldots, \alpha_n) \cong \langle \alpha_1, \ldots, \alpha_n \mid \emptyset \rangle .$$

In particular, $\mathbb{Z}$ has the presentation $\langle \alpha \mid \emptyset \rangle$.

(2) *The group $\mathbb{Z} \times \mathbb{Z}$ has the presentation $\langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle$.*

(3) *The cyclic group $\mathbb{Z}_n$ has the presentation*

$$\mathbb{Z}_n \cong \langle \alpha \mid \alpha^n = 1 \rangle .$$

(4) *The group $\mathbb{Z}_m \times \mathbb{Z}_n$ has the presentation*

$$\mathbb{Z}_m \times \mathbb{Z}_n \cong \langle \beta, \gamma \mid \beta^m = 1, \gamma^n = 1, \beta\gamma = \gamma\beta \rangle .$$

Proof. Part (3) follows from Theorem 178. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle$ be the isomorphism in (2). We have

$$\begin{aligned} \mathbb{Z}_m \times \mathbb{Z}_n &\cong (\mathbb{Z} \times \mathbb{Z})/(m\mathbb{Z} \times n\mathbb{Z}) \\ &\cong \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle /\varphi(m\mathbb{Z} \times n\mathbb{Z}). \\ &\cong \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle /\overline{\{\beta^m, \gamma^n\}} \\ &\cong \langle \beta, \gamma \mid \beta^m = 1, \gamma^n = 1, \beta\gamma = \gamma\beta \rangle \end{aligned}$$

by Theorem 178. □

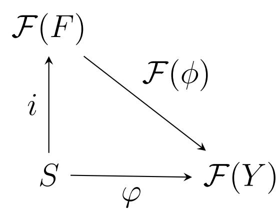
Theorem 180. *[Problem 9-7] The free abelian group on a set S is uniquely determined up to isomorphism by the characteristic property (Proposition 9.14).*

Proof. The proof is identical to that of Theorem 170. □

Theorem 181. *[Problem 9-8] Suppose G is a free abelian group of finite rank. Then every basis of G is finite.*

Proof. Let n be the rank of G. If G has an infinite basis then it has a linearly independent set S with $n+1$ elements, and $\mathbb{Z}S$ is a free abelian group of rank $n+1$ contained in G. This contradicts Proposition 9.19. □

Theorem 182. *[Problem 9-9] Suppose C is a concrete category with faithful functor $\mathcal{F} : \mathsf{C} \to \mathsf{Set}$. If S is a set, a free object on S in C is an object $F \in \mathrm{Ob}(\mathsf{C})$ together with a map $i : S \to \mathcal{F}(F)$, such that for any object $Y \in \mathrm{Ob}(\mathsf{C})$ and any map $\varphi : S \to \mathcal{F}(Y)$ there exists a unique morphism $\phi \in \mathrm{Hom}_{\mathsf{C}}(F, Y)$ such that the following diagram commutes:*



(1) *Any two free objects on the same set are isomorphic in C.*
(2) *A free group is a free object in Grp, and a free abelian group is a free object in Ab.*
(3) *The free objects in Top are discrete topological spaces.*

Proof. Let F_1 and F_2 be free objects on a set S with injections $i_1 : S \to \mathcal{F}(F_1)$ and $i_2 : S \to \mathcal{F}(F_2)$. By definition, there exist unique morphisms $\phi_1 : F_2 \to F_1$ and $\phi_2 : F_1 \to F_2$ such that $i_1 = \mathcal{F}(\phi_1) \circ i_2$ and $i_2 = \mathcal{F}(\phi_2) \circ i_1$. Now

$$\mathcal{F}(\phi_1) \circ \mathcal{F}(\phi_2) \circ i_1 = \mathcal{F}(\phi_1) \circ i_2 = i_1,$$

so $\mathcal{F}(\phi_1 \circ \phi_2) = \mathcal{F}(\phi_1) \circ \mathcal{F}(\phi_2) = \mathrm{Id}_{\mathcal{F}(F_1)}$ by uniqueness. Since $\mathcal{F}$ is faithful, this implies that $\phi_1 \circ \phi_2 = \mathrm{Id}_{F_1}$. Similarly, $\phi_2 \circ \phi_1 = \mathrm{Id}_{F_2}$. This proves (1). Part (2) follows from Theorem 169 and Theorem 171. Part (3) is obvious. □

Theorem 183. *[Exercise 10.9] A graph Γ is connected if and only if given any two vertices $v, v' \in \Gamma$, there is an edge path from v to v'. In a connected graph, any two vertices can be connected by a simple edge path.*

Proof. One direction is evident. Suppose that Γ is connected. Apply Theorem 107 with the equivalence relation on Γ defined by $x \sim y$ if and only if there is an edge path containing x and y; this proves the converse.

In a connected graph, we can connect any two vertices by an edge path. If the edge path has a vertex v appearing twice then it is of the form $(v_0, e_1, \ldots, v, \ldots, v, \ldots, e_k, v_k)$. Deleting all edges and vertices between the two appearances of v creates a shorter edge path between v_0 and v_k. By repeating the process we eventually create a simple edge path between v_0 and v_k. Note that if $v_0 = v_k$ then we can let the edge path be the trivial edge path (v_0). □

Theorem 184. *[Exercise 10.18] Let G be a group.*

(1) *$[G, G]$ is a normal subgroup of G.*
(2) *$[G, G]$ is trivial if and only if G is abelian.*
(3) *The quotient group $G/[G, G]$ is always abelian.*

Proof. First note that

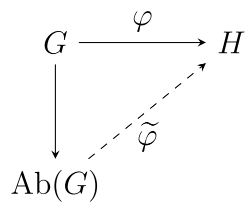
$$gaba^{-1}b^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}.$$

If $x \in [G, G]$ then $x = x_1 \cdots x_n$ for elements $x_i = a_i b_i a_i^{-1} b_i^{-1}$, so the above computation shows that $gxg^{-1} \in [G, G]$ and $g[G, G]g^{-1} \subseteq [G, G]$. Similarly we have $g^{-1}[G, G]g \subseteq [G, G]$, so $g[G, G]g^{-1} = [G, G]$. This proves (1). For (2), if G is abelian then $[G, G]$ is clearly trivial since $aba^{-1}b^{-1} = e$ for all $a, b \in G$. If $[G, G]$ is trivial and $a, b \in G$ then $aba^{-1}b^{-1} \in [G, G]$, so $aba^{-1}b^{-1} = e$ and therefore $ab = ba$. For (3), let $a[G, G]$ and $b[G, G]$ be elements of $G/[G, G]$. We have

$$ab[G, G] = baa^{-1}b^{-1}ab[G, G] = ba[G, G]$$

since $a^{-1}b^{-1}ab \in [G, G]$, which shows that $G/[G, G]$ is abelian. □

Theorem 185. *[Exercise 10.20] Let G be a group. For any abelian group H and any homomorphism $\varphi : G \to H$, there exists a unique homomorphism $\widetilde{\varphi} : \mathrm{Ab}(G) \to H$ such that the following diagram commutes:*



Proof. We have

$$\varphi(aba^{-1}b^{-1}) = \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = e$$

for all $a, b \in G$ since H is abelian. Therefore $[G,G] \subseteq \ker(\varphi)$ and there exists a unique homomorphism $\widetilde{\varphi} : \text{Ab}(G) \to H$ such that $\varphi = \widetilde{\varphi} \circ \pi$ where $\pi : G \to \text{Ab}(G)$ is the quotient map. □

Theorem 186. *[Problem 10-1]* $\mathbb{S}^n$ *is simply connected when* $n \geq 2$.

Proof. Let N be the north pole and let S be the south pole. Let $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$. Then $U \cup V = \mathbb{S}^n$, U and V are open in $\mathbb{S}^n$, and the sets U, V and $U \cap V$ are all path-connected (since $n \geq 2$). Therefore

$$\pi_1(\mathbb{S}^1) \cong (\pi_1(U) * \pi_1(V))/\overline{C}$$

by the Seifert-van Kampen theorem. But $U, V \approx \mathbb{R}^n$, so $\pi_1(U)$ and $\pi_1(V)$ are both trivial. This implies that $\pi_1(\mathbb{S}^1)$ is also trivial. □

Example 187. [Problem 10-2] Let $X \subseteq \mathbb{R}^3$ be the union of the unit 2-sphere with the line segment $\{0\} \times \{0\} \times [-1,1]$. Compute $\pi_1(X,N)$, where $N = (0,0,1)$ is the north pole, giving explicit generator(s).

Let $P = (1,0,0)$ and let B be a small coordinate ball around P in X. Let $U = X \setminus \{P\}$ so that $U \cup B = X$ and $U \cap B$ is homeomorphic to $\mathbb{S}^1$. By Corollary 10.5,

$$\pi_1(X,P) \cong \pi_1(U,P)/\overline{i_*\pi_1(U \cap B)}$$

where $i : U \cap B \to U$ is the canonical injection. But for any $[f] \in \pi_1(U \cap B)$ the loop $i \circ f$ is a loop in $\mathbb{S}^2 \setminus P$, which is simply connected. Therefore $i_*\pi_1(U \cap B)$ is trivial and $\pi_1(X,P) \cong \pi_1(U,P)$. But U is homotopy equivalent to $\mathbb{S}^1$, so $\pi_1(X,N) \cong \pi_1(X,P) \cong \mathbb{Z}$. A generator for $\pi_1(X,N)$ is the loop that traverses the line segment from the north pole down to the south pole and then returns to the north pole by a path on $\mathbb{S}^2$.

Theorem 188. *[Problem 10-3] Any two vertices in a tree are joined by a unique simple edge path.*

Proof. Suppose that v and w are two vertices in a tree that are joined by two different simple edge paths P and Q. Let $\overline{Q}$ be the edge path from w to v obtained by reversing Q, and let $P \cdot \overline{Q}$ be the edge path formed by concatenating P and $\overline{Q}$ and deleting the extra occurrence of w. By deleting duplicates from both ends, we can see that $P \cdot \overline{Q}$ contains a cycle. □

Theorem 189. *[Problem 10-4] Every vertex u in a finite tree T is a strong deformation retract of the tree.*

Proof. If the tree has no edges, there is nothing to prove. Suppose that u is incident with exactly one edge. The proof follows as in Theorem 10.10, but we use the fact that every tree with at least two vertices contains two vertices each incident with exactly one edge. By always choosing $v \neq u$ in the proof, we obtain a strong deformation retraction onto u. Now suppose that u is incident with two edges e, e' which have endpoints v, v' (not equal to u). We have $T \setminus (e \cup u \cup e') = U \cup U'$ where U and U' are disjoint trees containing v and v' respectively, and where v and v' are both incident with at most one edge. Choose strong deformation retractions from U to v and U' to v'. By combining these with the obvious strong deformation retraction from $\overline{e} \cup u \cup \overline{e'}$, we have a strong deformation retraction from T onto u. □

Theorem 190. *The fundamental groups of the following spaces are isomorphic to $\mathbb{Z}^{*n} = \mathbb{Z} * \cdots * \mathbb{Z}$:*

(1) $\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$, *the bouquet of n circles.*
(2) $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$, *the plane with n isolated points removed.*
(3) $\mathbb{S}^2 \setminus \{p_1, \ldots, p_{n+1}\}$, *the sphere with $n + 1$ isolated points removed.*

Proof. (1) follows from Theorem 10.7. (2) follows from the fact that $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ is homotopy equivalent to $\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$ (see Example 7.44). (3) follows from the fact that $\mathbb{S}^2 \setminus \{p_{n+1}\}$ is homeomorphic to $\mathbb{R}^2$. □

Example 191. [Problem 10-5] Compute the fundamental group of the complement of the three coordinate axes in $\mathbb{R}^3$, giving explicit generator(s).

Using the map $x \mapsto x/\|x\|$ we see that the space X is homeomorphic to the sphere $\mathbb{S}^2$ with the six points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ removed. By Theorem 190,

$$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

We can view X as $\mathbb{R}^3 \setminus \{0\}$ with six rays removed. Choose one of the six rays to exclude, and choose five loops that wind once around each of the remaining rays. Then these loops generate $\pi_1(X)$.

Theorem 192. *[Problem 10-6] Suppose M is a connected manifold of dimension at least 3, and $p \in M$. Then the inclusion $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism $\pi_1(M \setminus \{p\}) \cong \pi_1(M)$.*

Proof. Let B be a coordinate ball around p and let $U = B$ and $V = M\setminus\{p\}$ in Corollary 10.5. Choose some base point q in $B \setminus \{p\}$. Then the inclusion $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism

$$\pi_1(M, q) \cong \pi_1(M \setminus \{p\}, q)/\overline{j_*\pi_1(B \setminus \{p\}, q)}$$

where $j : B \setminus \{p\} \to M \setminus \{p\}$ is the inclusion. But $\pi_1(B \setminus \{p\}, q)$ is trivial by Corollary 7.38, so $\pi_1(M, q) \cong \pi_1(M \setminus \{p\}, q)$. □

Theorem 193. *[Problem 10-7] Suppose M_1 and M_2 are connected n-manifolds with $n \geq 3$. Then the fundamental group of $M_1\#M_2$ is isomorphic to $\pi_1(M_1) * \pi_1(M_2)$.*

Proof. By Theorem 92, there are open subsets $U_1, U_2 \subseteq M_1\#M_2$ and points $p_i \in M_i$ such that $U_i \approx M_i \setminus \{p_i\}$, $U_1 \cap U_2 \approx \mathbb{R}^n \setminus \{0\}$, and $U_1 \cup U_2 = M_1\#M_2$. Choose a base point q in $U_1 \cap U_2$. Since $\mathbb{R}^n \setminus \{0\}$ is simply connected for $n \geq 3$, by Corollary 10.4 and Theorem 192 we have

$$\begin{aligned}\pi_1(M_1\#M_2, q) &\cong \pi_1(U_1, q) * \pi_1(U_2, q)\\ &\cong \pi_1(M_1 \setminus \{p_1\}) * \pi_1(M_2 \setminus \{p_2\})\\ &\cong \pi_1(M_1) * \pi_1(M_2).\end{aligned}$$

□

Theorem 194. *[Problem 10-8] Suppose M_1 and M_2 are nonempty, compact, connected 2-manifolds. Then any two connected sums of M_1 and M_2 are homeomorphic.*

Proof. Since any connected sum $M_1\#M_2$ is also a nonempty, compact, connected 2-manifold, it follows from Theorem 10.22 that it suffices to prove that any two connected sums have isomorphic fundamental groups. This follows from 92. □

Example 195. [Problem 10-9] Let X_n be the union of the n circles of radius 1 that are centered at the points $\{0, 2, 4, \ldots, 2n-2\}$ in $\mathbb{C}$, which are pairwise tangent to each other along the x-axis. Prove that $\pi_1(X_n, 1)$ is a free group on n generators, and describe explicit loops representing the generators.

This is identical to Example 10.8 - we have $\pi_1(X_n, 1) \cong \mathbb{Z} * \cdots * \mathbb{Z}$. For each $k = 0, \ldots, n-1$, let ω_k be a loop based at $(2k-1, 0)$ that traverses the $(k+1)$th circle once, and let γ_k be a path from 1 to $(2k-1, 0)$. Then $\{\gamma_k \cdot \omega_k \cdot \overline{\gamma_k} : k = 0, \ldots, n-1\}$ are n loops representing the generators of $\pi_1(X_n, 1)$.

Theorem 196. *[Problem 10-10] For any finitely presented group G, there is a finite CW complex whose fundamental group is isomorphic to G.*

Proof. Let $\langle \alpha_1, \ldots, \alpha_n \mid r_1, \ldots, r_m\rangle$ be a presentation of G. Let Γ be a graph with a single vertex v and n loops at v; denote these loops by $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n$. By Theorem 10.12, $\pi_1(\Gamma, v) \cong \langle \alpha_1, \ldots, \alpha_n \mid \emptyset\rangle$. Let $\Gamma_0 = \Gamma$. For each $i = 1, \ldots, m$, write $r_i = \alpha_{k_1}^{p_1} \cdots \alpha_{k_\ell}^{p_\ell}$, let $\beta : I \to \Gamma$ be a representative of $[\widetilde{\alpha}_{k_1}^{p_1}] \cdots [\widetilde{\alpha}_{k_\ell}^{p_\ell}] \in \pi_1(\Gamma, v)$ and let $\widetilde{\beta} : \mathbb{S}^1 \to \Gamma$ be the circle representative of β. Attach a 2-cell $D_i = \overline{\mathbb{B}^2}$ to Γ_{i-1} along the attaching map $\widetilde{\beta}$ to obtain a new CW complex Γ_i; then

$$\pi_1(\Gamma_i, v) \cong \langle \alpha_1, \ldots, \alpha_n \mid r_1, \ldots, r_i \rangle$$

by Proposition 10.13. By repeating the process, we have $\pi_1(\Gamma_m, v) \cong G$. □

Example 197. [Problem 10-11] For each of the following spaces, give a presentation of the fundamental group together with a specific loop representing each generator.

(1) A closed disk with two interior points removed.
(2) The projective plane with two points removed.
(3) A connected sum of n tori with one point removed.
(4) A connected sum of n tori with two points removed.

Denote these spaces by $X_1, \ldots, X_4$.

(1) $\pi_1(X_1) \cong \langle \alpha, \beta \mid \emptyset \rangle$ where α is a loop around the first interior point and β is a loop around the second interior point.

Example 198. [Problem 10-12] Give a purely algebraic proof that the groups $\langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} \rangle$ and $\langle \rho, \gamma \mid \rho^2\gamma^2 \rangle$ are isomorphic.

This follows from Lemma 6.16.

Theorem 199. *[Problem 10-16] Abelianization defines a functor from* Grp *to* Ab*.*

Proof. For any two groups G, H and any homomorphism $f : G \to H$, define $\text{Ab}(f) : \text{Ab}(G) \to \text{Ab}(H)$ as the unique homomorphism satisfying $\text{Ab}(f) \circ \pi_G = \pi_H \circ f$, where $\pi_G : G \to \text{Ab}(G)$ and $\pi_H : H \to \text{Ab}(H)$ are the quotient maps. If $G = H$ and $f = \text{Id}_G$ then $\text{Ab}(f) \circ \pi = \pi$, so $\text{Ab}(f) = \text{Id}_{\text{Ab}(G)}$ by uniqueness. If K is a group and $g : H \to K$ is a homomorphism then

$$\begin{aligned} \text{Ab}(g \circ f) \circ \pi_G &= \pi_K \circ g \circ f \\ &= \text{Ab}(g) \circ \pi_H \circ f \\ &= \text{Ab}(g) \circ \text{Ab}(f) \circ \pi_G, \end{aligned}$$

so $\text{Ab}(g \circ f) = \text{Ab}(g) \circ \text{Ab}(f)$ by uniqueness. This shows that $\text{Ab} : \mathsf{Grp} \to \mathsf{Ab}$ is a functor. □

Theorem 200. *[Problem 10-17]* $\text{Ab}(G)$ *is the unique group (up to isomorphism) that satisfies the characteristic property expressed in Theorem 10.19, for any group* G*.*

Proof. Suppose $\mathrm{Ab}(G)$ and $\mathrm{Ab}(G)'$ are two groups that satisfy the characteristic property. Let $\pi : G \to \mathrm{Ab}(G)$ and $\pi' : G \to \mathrm{Ab}(G)'$ be the canonical maps. There exist unique homomorphisms $\varphi : \mathrm{Ab}(G)' \to \mathrm{Ab}(G)$ and $\varphi' : \mathrm{Ab}(G) \to \mathrm{Ab}(G)'$ satisfying $\varphi \circ \pi' = \pi$ and $\varphi' \circ \pi = \pi'$. Since $\varphi \circ \varphi' \circ \pi = \varphi \circ \pi' = \pi$ and $\varphi' \circ \varphi \circ \pi' = \varphi' \circ \pi = \pi'$, we have $\varphi \circ \varphi' = \mathrm{Id}_{\mathrm{Ab}(G)}$ and $\varphi' \circ \varphi = \mathrm{Id}_{\mathrm{Ab}(G)'}$ by uniqueness. Therefore $\mathrm{Ab}(G) \cong \mathrm{Ab}(G)'$. ☐

Theorem 201. *[Problem 10-18] For any groups G_1 and G_2,*

$$\mathrm{Ab}(G_1 * G_2) \cong \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2).$$

Proof. For $i = 1, 2$, let

$$\begin{aligned} \alpha_i &: G_i \to \mathrm{Ab}(G_i), \\ \alpha &: G_1 * G_2 \to \mathrm{Ab}(G_1 * G_2), \\ j_i &: \mathrm{Ab}(G_i) \to \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2), \\ k_i &: G_i \to G_1 * G_2 \end{aligned}$$

be the canonical maps. There exists a unique homomorphism $\ell : G_1 * G_2 \to \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$ satisfying $\ell \circ k_i = j_i \circ \alpha_i$, and there exists a unique homomorphism $\varphi : \mathrm{Ab}(G_1 * G_2) \to \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$ satisfying $\varphi \circ \alpha = \ell$. Also, there exist unique homomorphisms $m_i : \mathrm{Ab}(G_i) \to \mathrm{Ab}(G_1 * G_2)$ satisfying $m_i \circ \alpha_i = \alpha \circ k_i$, so there exists a unique homomorphism $\psi : \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2) \to \mathrm{Ab}(G_1 * G_2)$ satisfying $\psi \circ j_i = m_i$. Now

$$\varphi \circ \psi \circ j_i \circ \alpha_i = \varphi \circ m_i \circ \alpha_i = \varphi \circ \alpha \circ k_i = \ell \circ k_i = j_i \circ \alpha_i,$$

so $\varphi \circ \psi = \mathrm{Id}_{\mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)}$ by uniqueness. Similarly,

$$\psi \circ \varphi \circ \alpha \circ k_i = \psi \circ \ell \circ k_i = \psi \circ j_i \circ \alpha_i = m_i \circ \alpha_i = \alpha \circ k_i,$$

so $\psi \circ \varphi = \mathrm{Id}_{\mathrm{Ab}(G_1 * G_2)}$ by uniqueness. ☐

Corollary 202. *The abelianization of a free group on n generators is free abelian of rank n, and isomorphic finitely generated free groups have the same number of generators.*

Theorem 203. *[Problem 10-19] For any set S, the abelianization of the free group $F(S)$ is isomorphic to the free abelian group $\mathbb{Z}S$.*

Proof. Let

$$\begin{aligned} j &: S \to \mathbb{Z}S, \\ k &: S \to F(S), \\ \ell &: F(S) \to \mathrm{Ab}(F(S)) \end{aligned}$$

be the canonical maps. There exist unique homomorphisms $\varphi : \mathbb{Z}S \to \mathrm{Ab}(F(S))$ and $\psi : \mathrm{Ab}(F(S)) \to \mathbb{Z}S$ such that $\varphi \circ j = \ell \circ k$ and $\psi \circ \ell \circ k = j$. Then

$$\varphi \circ \psi \circ \ell \circ k = \varphi \circ j = \ell \circ k,$$

so $\varphi \circ \psi = \mathrm{Id}_{\mathrm{Ab}(F(S))}$ by uniqueness. Similarly,

$$\psi \circ \varphi \circ j = \psi \circ \ell \circ k = j,$$

so $\psi \circ \varphi = \mathrm{Id}_{\mathbb{Z}S}$ by uniqueness. □

Theorem 204. *[Problem 10-20] Let Γ be a finite connected graph. The Euler characteristic of Γ is $\chi(\Gamma) = V - E$, where V is the number of vertices and E is the number of edges. The fundamental group of Γ is a free group on $1-\chi(\Gamma)$ generators, and therefore $\chi(\Gamma)$ is a homotopy invariant.*

Proof. Let T be a spanning tree in Γ. Since the number of vertices in a tree is always one more than the number of edges, we have $\chi(T) = 1$. By Theorem 10.12, there is one generator of $\pi_1(\Gamma)$ for each edge of Γ not in T. Therefore $\chi(\Gamma) = 1 - n$ where n is the number of generators of $\pi_1(\Gamma)$, and the result follows. □

Theorem 205. *[Problem 10-21]*

(1) *If a pushout of a pair of morphisms exists, it is unique up to isomorphism in the category* C.
(2) *The amalgamated free product is the pushout of two group homomorphisms with the same domain.*
(3) *Let S_1 and S_2 be sets with nonempty intersection. In the category of sets, the pushout of the inclusions $S_1 \cap S_2 \hookrightarrow S_1$ and $S_1 \cap S_2 \hookrightarrow S_2$ is $S_1 \cup S_2$ together with appropriate inclusion maps.*
(4) *Suppose X and Y are topological spaces, $A \subseteq Y$ is a closed subset, and $f : A \to X$ is a continuous map. The adjunction space $X \cup_f Y$ is the pushout of (ι_A, f) in the category* Top.
(5) *In the category* Top, *given two continuous maps with the same domain, the pushout always exists.*

Proof. Let $f_i : A_0 \to A_i$ for $i = 1, 2$ be a pair of morphisms. Suppose there are two pushouts, P and P', with morphisms $g_i : A_i \to P$ and $g_i' : A_i \to P'$. Then there exist unique morphisms $h : P' \to P$ and $h' : P \to P'$ such that $h \circ g_i' = g_i$ and $h' \circ g_i = g_i'$. Since

$$h \circ h' \circ g_i = h \circ g_i' = g_i \quad \text{and} \quad h' \circ h \circ g_i' = h' \circ g_i = g_i',$$

we have $h \circ h' = \mathrm{Id}_P$ and $h' \circ h = \mathrm{Id}_{P'}$ by uniqueness. This proves (1).

Let $f_i : A_0 \to A_i$ be a pair of group homomorphisms. We want to show that $A_1 *_{A_0} A_2$ is the pushout of A. Let $j_i : A_i \to A_1 * A_2$ and $\pi : A_1 * A_2 \to A_1 *_{A_0} A_2$ be the usual inclusions, and choose $g_i = \pi \circ j_i$. Let

$$C = \left\{ (j_1 \circ f_1)(a)(j_2 \circ f_2)(a)^{-1} : a \in A_0 \right\}$$

so that $A_1 *_{A_0} A_2 = (A_1 * A_2)/\overline{C}$. If $a \in A_0$ then

$$\begin{aligned}(g_1 \circ f_1)(a) &= (j_1 \circ f_1)(a)\overline{C} \\ &= (j_1 \circ f_1)(a)(j_1 \circ f_1)(a)^{-1}(j_2 \circ f_2)(a)\overline{C} \\ &= (j_2 \circ f_2)(a)\overline{C} \\ &= (g_2 \circ f_2)(a),\end{aligned}$$

which shows that $g_1 \circ f_1 = g_2 \circ f_2$. Now let B be a group and let $h_i : A_i \to B$ be a pair of homomorphisms such that $h_1 \circ f_1 = h_2 \circ f_2$. There exists a unique homomorphism $\widehat{h} : A_1 * A_2 \to B$ such that $\widehat{h} \circ j_i = h_i$. For $a \in A_0$ we have

$$\begin{aligned}\widehat{h}((j_1 \circ f_1)(a)(j_2 \circ f_2)(a)^{-1}) &= (\widehat{h} \circ j_1 \circ f_1)(a)(\widehat{h} \circ j_2 \circ f_2)(a)^{-1} \\ &= (h_1 \circ f_1)(a)(h_2 \circ f_2)(a)^{-1} \\ &= e,\end{aligned}$$

so $\overline{C} \subseteq \ker(\widehat{h})$ and there exists a unique homomorphism $h : A_1 *_{A_0} A_2 \to B$ such that $h \circ \pi = \widehat{h}$. Since

$$h \circ g_i = h \circ \pi \circ j_i = \widehat{h} \circ j_i = h_i,$$

this proves (2).

For (3), let $f_i : S_i \to T$ be a pair of functions such that $f_1|_{S_1 \cap S_2} = f_2|_{S_1 \cap S_2}$. Define $g : S_1 \cup S_2 \to T$ by letting $g|_{S_i} = f_i$; this is well-defined due to the conditions on f_1 and f_2. For (4), let $g_1 : X \to Z$ and $g_2 : Y \to Z$ be continuous maps such that $g_1 \circ f = g_2|_A$. Let $j_1 : X \to X \amalg Y$ and $j_2 : Y \to X \amalg Y$ be the canonical injections. Let $\widehat{g} : X \amalg Y \to Z$ be the unique continuous map such that $\widehat{g} \circ j_1 = g_1$ and $\widehat{g} \circ j_2 = g_2$; then $\widehat{g}$ descends to a unique continuous map $g : X \cup_f Y \to Z$ satisfying $g \circ q = \widehat{g}$ where $q : X \amalg Y \to X \cup_f Y$ is the quotient map. Since $g \circ (q \circ j_i) = \widehat{g} \circ j_i = g_i$ for $i = 1, 2$, this proves (4). For (5), let $f_i : A_0 \to A_i$ be a pair of continuous maps and define a relation $\sim$ on $A_1 \amalg A_2$ as the smallest equivalence relation such that $f_1(x) \sim f_2(x)$ for every $x \in A_0$. Take the pushout P to be the quotient space $(A_1 \amalg A_2)/\sim$; the proof is then similar to (4). ☐

Chapter 11. Covering Maps

Theorem 206. *[Exercise 11.2]*

1. *Every covering map is a local homeomorphism, an open map, and a quotient map.*
2. *An injective covering map is a homeomorphism.*
3. *A finite product of covering maps is a covering map.*
4. *The restriction of a covering map to a saturated, connected, open subset is a covering map onto its image.*

Proof. (1) and (2) follow from Theorem 13 and Proposition 3.69. Part (3) is obvious. For (4), let $q : E \to X$ be a covering map and let F be a saturated, connected, open subset of E. If $x \in q(F)$ then x has some evenly covered neighborhood $U \subseteq X$. Then $(q|_F)^{-1}(U \cap q(F)) = q^{-1}(U \cap q(F))$ since F is saturated, which shows that $q|_F : F \to q(F)$ is a covering map. □

Example 207. [Exercise 11.7] Let X_n be the union of n circles in $\mathbb{C}$ as described in Example 195. Define a map $q : X_3 \to X_2$ by letting A, B, and C denote the unit circles centered at 0, 2, and 4, respectively, and defining

$$q(z) = \begin{cases} z, & z \in A; \\ 2 - (z-2)^2, & z \in B; \\ 4 - z, & z \in C. \end{cases}$$

Show that q is a covering map.

Let $\delta > \varepsilon > 0$ be small numbers; then $X_2 \cap B_\delta(1)$ and $X_2 \setminus \overline{B}_\varepsilon(1)$ are evenly covered open sets whose union is X_2.

Example 208. [Exercise 11.9] Let E be the interval $(0, 2) \subseteq \mathbb{R}$, and define $f : E \to \mathbb{S}^1$ by $f(x) = e^{2\pi i x}$. Then f is a local homeomorphism and is clearly surjective, but f is not a covering map since the point $1 \in \mathbb{S}^1$ has no evenly covered neighborhood.

If a small neighborhood U around 1 is chosen then the connected components of $f^{-1}(U)$ do not map onto U; if U is a large neighborhood around 1 then the connected components of $f^{-1}(U)$ do not map injectively into U.

Theorem 209. *[Exercise 11.25] If S_1 and S_2 are right G-sets and $\varphi : S_1 \to S_2$ is a G-isomorphism, then φ^{-1} is also a G-isomorphism.*

Proof. Let $s_2 \in S_2$ and $g \in G$. Then $\varphi^{-1}(s_2 \cdot g) = \varphi^{-1}(\varphi(\varphi^{-1}(s_2) \cdot g)) = \varphi^{-1}(s_2) \cdot g$, which shows that φ^{-1} is G-equivariant. □

Theorem 210. *Let $q : E \to X$ be a covering map and let $f : X \to Y$ be any function. Then f is continuous if and only if $f \circ q$ is continuous.*

Proof. One direction is evident. Suppose that $f \circ q$ is continuous. Let $x \in X$, let U be an evenly covered neighborhood of x and let $\sigma : U \to E$ be a local section of q so that $q \circ \sigma = \mathrm{Id}_U$. Then $f|_U = (f|_U \circ q) \circ \sigma$ which is continuous since $f \circ q$ is continuous. By Proposition 2.19, f is continuous. □

Theorem 211. *[Problem 11-1] Suppose $q : E \to X$ is a covering map.*

(1) *If X is Hausdorff then E is too.*
(2) *If X is an n-manifold then E is too.*
(3) *If E is an n-manifold and X is Hausdorff then X is an n-manifold.*

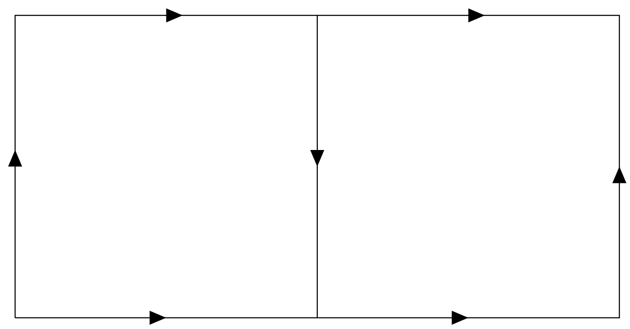


FIGURE 0.2. A covering of the Klein bottle by the torus.

Proof. For (1), let x and y be distinct points in E. Let U be an evenly covered neighborhood of $q(x)$ and let $\widetilde{U}$ be the sheet of $q^{-1}(U)$ containing x so that $q|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. If $y \in \widetilde{U}$ then we are done since $\widetilde{U}$ is Hausdorff, and if $y \notin \widetilde{U}$ then $\widetilde{U}$ is a neighborhood of x and $E \setminus \widetilde{U}$ is a neighborhood of y. Part (2) follows from Proposition 4.40 of [1]. Part (3) follows from Proposition 3.56. $\square$

Theorem 212. *[Problem 11-2] For any $n \geq 1$, the map $q : \mathbb{S}^n \to \mathbb{P}^n$ defined in Example 11.6 is a covering map.*

Proof. It is clear that q is continuous and surjective. For each $i = 1, \ldots, n$, let $\pi_i : \mathbb{S}^n \to \mathbb{R}$ be the projection onto the ith coordinate, let $\mathbb{S}^n_{i+} = \pi_i^{-1}((0, \infty))$ and let $\mathbb{S}^n_{i-} = \pi_i^{-1}((-\infty, 0))$. Let $\mathbb{P}^n_i = \mathbb{P}^n \setminus q(\pi_i^{-1}(\{0\}))$; then $q^{-1}(\mathbb{P}^n_i)$ is the disjoint union of $\mathbb{S}^n_{i+}$ and $\mathbb{S}^n_{i-}$ while $q|_{\mathbb{S}^n_{i+}}$ and $q|_{\mathbb{S}^n_{i-}}$ are homeomorphisms. Since the sets $\mathbb{P}^n_1, \ldots, \mathbb{P}^n_n$ cover $\mathbb{P}^n$, this shows that q is a covering map. $\square$

Example 213. [Problem 11-3] Let S be the following subset of $\mathbb{C}^2$:

$$S = \left\{(z, w) : w^2 = z, w \neq 0\right\}.$$

(It is the graph of the two-valued complex square root "function" described in Chapter 1, with the origin removed.) Show that the projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ onto the first coordinate restricts to a two-sheeted covering map $q : S \to \mathbb{C} \setminus \{0\}$.

Let R be any ray extending from (and including) the origin; then $q^{-1}(\mathbb{C} \setminus R)$ consists of two sheets. The sets $\mathbb{C} \setminus R$ cover $\mathbb{C} \setminus \{0\}$ as R varies, which shows that q is a covering map.

Example 214. [Problem 11-4] Show that there is a two-sheeted covering of the Klein bottle by the torus.

See Figure 0.2.

Theorem 215. *[Problem 11-5] Let M and N be connected manifolds of dimension $n \geq 2$, and suppose $q : \widetilde{M} \to M$ is a k-sheeted covering map. Then there is a connected sum $M \# N$ that admits a k-sheeted covering by a manifold of the form $\widetilde{M} \# N \cdots \# N$ (connected sum of $\widetilde{M}$ with k disjoint copies of N).*

Proof. Let U be some evenly covered neighborhood in M and let B be a regular coordinate ball that lies inside U. Then $q^{-1}(B)$ is the disjoint union of k coordinate balls $\widetilde{B}_1, \ldots, \widetilde{B}_k$ in $\widetilde{M}$. Choose a regular coordinate ball C in N. Form the connected sum $M \# N$ by identifying ∂B with ∂C, and form the connected sum $\widetilde{M} \# N \cdots \# N$ by identifying $\widetilde{B}_i$ with C_i for $i = 1, \ldots, k$, where $C_i \subseteq N_i$ is C in the ith copy N_i of N. Define a covering map $q' : \widetilde{M} \# N \cdots \# N \to M \# N$ by setting $q'|_{\widetilde{M} \setminus \widetilde{B}_i} = q|_{\widetilde{M} \setminus \widetilde{B}_i}$ and $q'|_{N_i \setminus C_i} = \mathrm{Id}_{N \setminus C}$ for each i. □

Theorem 216. *[Problem 11-6] Every nonorientable compact surface of genus $n \geq 1$ has a two-sheeted covering by an orientable one of genus $n - 1$.*

Proof. If X is a surface, write $X^{\# k}$ for $X \# \cdots \# X$, the connected sum of k copies of X. We use induction on n. The case $n = 1$ follows from Theorem 212, and the case $n = 2$ follows from Example 214. Assume that the result holds for $n = 2, \ldots, k$, and consider a nonorientable compact surface $(\mathbb{P}^2)^{\# k+1}$. By the induction hypothesis, there is a two-sheeted covering map $q : (\mathbb{T}^2)^{\# k-2} \to (\mathbb{P}^2)^{\# k-1}$. By Theorem 215, there exists a two-sheeted covering map $q' : (\mathbb{T}^2)^{\# k} \to (\mathbb{P}^2)^{\# k-1} \# \mathbb{T}^2$. But Lemma 6.17 shows that $\mathbb{T}^2 \# \mathbb{P}^2$ is homeomorphic to $(\mathbb{P}^2)^{\# 3}$, so there is a covering map $q'' : (\mathbb{T}^2)^{\# k} \to (\mathbb{P}^2)^{\# k+1}$. □

Theorem 217. *[Problem 11-9] Every proper local homeomorphism between connected, locally path-connected and compactly generated Hausdorff spaces is a covering map.*

Proof. Let $q : X \to Y$ be such a map and let $y \in Y$. Since $\{y\}$ is compact and q is proper, $q^{-1}(\{y\})$ is also compact. This implies that $q^{-1}(\{y\})$ is a finite discrete set since q is a local homeomorphism. Write $q^{-1}(\{y\}) = \{p_1, \ldots, p_n\}$; by Lemma 109, we can choose pairwise disjoint open sets $U_1, \ldots, U_n$ with $p_i \in U_i$. By shrinking each set, we can assume that $q|_{U_i} : U_i \to q(U_i)$ is a homeomorphism for $i = 1, \ldots, n$. Let $U = \bigcup_{i=1}^n U_i$. By Theorem 4.95, $q(X \setminus U)$ is closed in Y, and $V = \bigcap_{i=1}^n q(U_i) \cap (Y \setminus q(X \setminus U))$ is a neighborhood of y that is evenly covered: it is clear that V is open, and if $q(x) \in V$ then $q(x) \notin q(X \setminus U)$, so $x \in U$. Therefore $q^{-1}(V)$ is the disjoint union of the open sets $q^{-1}(V) \cap U_i$, each of which is mapped homeomorphically onto V. □

Theorem 218. *[Problem 11-10] A covering map is proper if and only if it is finite-sheeted.*

Proof. Every covering map is a local homeomorphism, so one direction follows from the argument used in Theorem 217. Conversely, suppose that the covering map $q : E \to X$

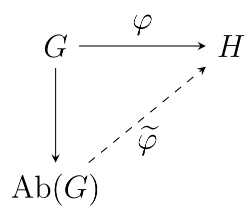
is finite-sheeted, let $B \subseteq X$ be a compact set, and let $\mathcal{U}$ be an open cover of $q^{-1}(B)$. For each $x \in B$ the set $q^{-1}(\{x\})$ is finite. Write $q^{-1}(\{x\}) = \{\widetilde{x}_1, \dots, \widetilde{x}_k\}$ and let W be an evenly covered neighborhood of x so that $q^{-1}(W) = \widetilde{W}_1 \cup \cdots \cup \widetilde{W}_k$ where each $\widetilde{W}_i$ is connected and open, $q|_{\widetilde{W}_i} : \widetilde{W}_i \to W$ is a homeomorphism, and $\widetilde{x}_i \in \widetilde{W}_i$. For each i choose a set $U_i \in \mathcal{U}$ containing $\widetilde{x}_i$. Let $V_x = \bigcap_{i=1}^k q(U_i \cap \widetilde{W}_i)$; then V_x is a neighborhood of x and $q^{-1}(V_x)$ is covered by $U_1, \dots, U_n$. Since B is compact, there are finitely many such sets $V_{x_1}, \dots, V_{x_n}$ that cover B. Since $q^{-1}(B)$ is covered by $q^{-1}(V_{x_1}), \dots, q^{-1}(V_{x_n})$ and each $q^{-1}(V_{x_i})$ is covered by finitely many sets in $\mathcal{U}$, this shows that $q^{-1}(B)$ is compact. ☐

Theorem 219. *[Problem 11-11] Let $q : E \to X$ be a covering map. Then E is compact if and only if X is compact and q is a finite-sheeted covering.*

Proof. If E is compact then X is compact. If $x \in X$ then $q^{-1}(\{x\})$ is closed and therefore compact. Since q is a local homeomorphism, $q^{-1}(\{x\})$ must be a finite discrete set. The converse follows from Theorem 218. ☐

Theorem 220. *[Problem 11-12] A continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be **odd** if $f(-z) = -f(z)$ for all $z \in \mathbb{S}^1$, and **even** if $f(z) = f(-z)$ for all $z \in \mathbb{S}^1$.*

(1) *Let $p_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be the two-sheeted covering map of Example 11.4. If f is odd, there exists a continuous map $g : \mathbb{S}^1 \to \mathbb{S}^1$ with $\deg f = \deg g$ such that the following diagram commutes:*



(2) *If $\deg f$ is also even, then g lifts to a map $\widetilde{g} : \mathbb{S}^1 \to \mathbb{S}^1$ such that $p_2 \circ \widetilde{g} = g$. Furthermore, $\widetilde{g} \circ p_2$ and f are both lifts of $g \circ p_2$ that agree at either 1 or -1, so they are equal everywhere.*
(3) *Every odd map has odd degree.*

Proof. Suppose that f is odd and define $g : \mathbb{S}^1 \to \mathbb{S}^1$ by $g(z) = f(\sqrt{z})^2$. We have $g(z^2) = f(|z|)^2 = f(z)^2$ for all $z \in \mathbb{S}^1$ since $f(-z)^2 = (-f(z))^2 = f(z)^2$, so $g \circ p_2 = p_2 \circ f$. The continuity of g follows from Theorem 210, and $\deg f = \deg g$ by Proposition 8.15. This proves (1). Choose a point $e \in \mathbb{S}^1$ such that $p_2(e) = g(1)$. The existence of $\widetilde{g}$ in (2) follows from Theorem 11.18, for if we identify $\pi_1(\mathbb{S}^1, g(1))$ with $\mathbb{Z}$ then $(p_2)_*\pi_1(\mathbb{S}^1, e) = 2\mathbb{Z}$ and $g_*\pi_1(\mathbb{S}^1, 1) = n\mathbb{Z} \subseteq 2\mathbb{Z}$ since n is even. We have $p_2 \circ \widetilde{g} \circ p_2 = g \circ p_2 = p_2 \circ f$, so $\widetilde{g} \circ p_2$ and f are both lifts of $g \circ p_2$. Also, since $\widetilde{g}(1)^2 = g(1) = f(1)^2$ we have

$(\widetilde{g}\circ p_2)(1) = (\widetilde{g}\circ p_2)(-1) = \pm f(1)$, so $\widetilde{g}\circ p_2$ and f agree at either 1 or -1. By Theorem 11.12, $\widetilde{g}\circ p_2 = f$. This proves (2). But f is odd while $\widetilde{g}\circ p_2$ is even, which is a contradiction. This proves (3). □

Theorem 221. *[Problem 11-13] Every even map $f : \mathbb{S}^1 \to \mathbb{S}^1$ has even degree.*

Proof. Define $g : \mathbb{S}^1 \to \mathbb{S}^1$ by $g(z) = f(\sqrt{z})$. Then $g(z^2) = f(|z|) = f(z)$ for all $z \in \mathbb{S}^1$ since f is even, and the continuity of g follows from Theorem 210. Now $\deg(f) = \deg(g \circ p_2) = 2\deg(g)$, so $\deg(f)$ is even. □

Theorem 222. *[Problem 11-14] For any continuous map $F : \mathbb{S}^2 \to \mathbb{R}^2$, there is a point $x \in \mathbb{S}^2$ such that $F(x) = F(-x)$.*

Proof. Suppose that $F(x) \neq F(-x)$ for all $x \in \mathbb{S}^2$ and define a continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ by

$$f(x) = \frac{F(x) - F(-x)}{\|F(x) - F(-x)\|}.$$

Let $g : \mathbb{S}^1 \times I \to \mathbb{S}^2$ be given by

$$((x, y), t) \mapsto \left(\sqrt{1-t^2}x, \sqrt{1-t^2}y, t\right);$$

then $f \circ g$ is a homotopy from $f|_{\mathbb{S}^1}$ to a constant map. But $f|_{\mathbb{S}^1}$ is odd and has odd degree by Theorem 220, so $f|_{\mathbb{S}^1}$ cannot be null-homotopic. □

Theorem 223. *[Problem 11-15] Given three disjoint, bounded, connected open subsets $U_1, U_2, U_3 \subseteq \mathbb{R}^3$, there exists a plane that simultaneously bisects all three, in the sense that the plane divides $\mathbb{R}^3$ into two half-spaces H^+ and H^- such that for each i, $U_i \cap H^+$ has the same volume as $U_i \cap H^-$.*

Proof. We first show that for any $x \in \mathbb{S}^2$ and any open set $U \subseteq \mathbb{R}^3$, there is a real number λ such that the plane through λx and orthogonal to x bisects U. Fix some i and define $V : \mathbb{R} \to [0, \infty)$ by $\lambda \mapsto \mathrm{Vol}(U \cap H^+_\lambda)$ where $H^+_\lambda = \{v \in \mathbb{R}^3 : \langle x, v\rangle > \langle x, \lambda x\rangle\}$. Since V is monotonic and continuous and $0, \mathrm{Vol}(U) \in V(\mathbb{R})$, there exists a unique $\lambda \in \mathbb{R}$ such that $V(\lambda) = \mathrm{Vol}(U)/2$. For $x \in \mathbb{S}^2$ and an open set $U \subseteq \mathbb{R}^3$, denote this value of λ by $\Lambda_U(x)$. Note that $\Lambda_U(-x) = -\Lambda_U(x)$. Define $F : \mathbb{S}^2 \to \mathbb{R}^2$ by $x \mapsto (\Lambda_{U_1}(x) - \Lambda_{U_2}(x), \Lambda_{U_2}(x) - \Lambda_{U_3}(x))$; by Theorem 222 there exists some $x \in \mathbb{S}^2$ such that $F(x) = F(-x)$, i.e. $\Lambda_{U_1}(x) = \Lambda_{U_2}(x) = \Lambda_{U_3}(x)$. □

Example 224. [Problem 11-16] Let T be the topologist's sine curve (Example 4.17), and let Y be the union of T with a semicircular arc that intersects T only at $(0, 1)$ and $(2/\pi, 1)$.

(1) Show that Y is simply connected.
(2) Show that there is a continuous map $f : Y \to \mathbb{S}^1$ that has no lift to $\mathbb{R}$.

From Example 86 we know that Y has exactly two path components, each of which is simply connected. Therefore Y is simply connected. Let A be the semicircular arc, let $\gamma : I \to A$ be a path that traces out A from $(0,1)$ to $(2/\pi, 1)$ and let $\omega : [0,1] \to \mathbb{S}^1$ be the loop $s \mapsto e^{2\pi i s}$. Let $g : T \to \mathbb{S}^1$ be the constant map $x \mapsto 1$. By the gluing lemma, we can construct a continuous map $f : Y \to \mathbb{S}^1$ such that $f|_T = g$ and $f|_A = \omega \circ \gamma^{-1}$. If $\widetilde{f} : Y \to \mathbb{R}$ is a lift of f then $\widetilde{f}((0,1)) = \widetilde{f}((2/\pi, 1))$, but this is impossible for $\widetilde{f}|_A \circ \gamma$ would be a lift of a loop with winding number 1 while $(\widetilde{f}|_A \circ \gamma)(0) = (\widetilde{f}|_A \circ \gamma)(1)$.

Theorem 225. *[Problem 11-18] If X is a topological space that has a universal covering space then X is semilocally simply connected.*

Proof. Let $q : E \to X$ be a covering map with E simply connected. Let $x \in X$, let U be an evenly covered neighborhood of x, choose some $\widetilde{x} \in q^{-1}(\{x\})$ and let $\widetilde{U}$ be the component of $q^{-1}(U)$ containing $\widetilde{x}$. If $\gamma : I \to U$ is a loop based at x then by Corollary 11.14 there is a path $\widetilde{\gamma} : I \to E$ such that $\gamma = q \circ \widetilde{\gamma}$ and $\widetilde{\gamma}(0) = \widetilde{x}$. But $\widetilde{\gamma}(I) \subseteq q^{-1}(U)$ is connected, so $\widetilde{\gamma}(I) \subseteq \widetilde{U}$; the fact that $\widetilde{\gamma}(1) \in q^{-1}(\{x\})$ implies that $\widetilde{\gamma}(1) = \widetilde{x}$. Since E is simply connected, $\widetilde{\gamma}$ is null-homotopic in E and therefore γ is null-homotopic in X. This shows that U is relatively simply connected. □

Example 226. [Problem 11-19] For each $n \in \mathbb{N}$, let C_n denote the circle in $\mathbb{R}^2$ with center $(1/n, 0)$ and radius $1/n$. The **Hawaiian earring** is the space $H = \bigcup_{n \in \mathbb{N}} C_n$, with the subspace topology.

(1) H is not semilocally simply connected, and therefore has no universal covering space.
(2) The cone on H is simply connected and semilocally simply connected, but not locally simply connected.

The point $(0,0) \in H$ does not have a relatively simply connected neighborhood, for any neighborhood of $(0,0)$ must contain some circle C_n for n sufficiently large. The cone $CH = (H \times I)/(H \times \{0\})$ on H is contractible and is therefore simply connected and semilocally simply connected. Consider the open set $U = H \times (1/2, 1]$ as a subset of CH; the point $(0,0,1)$ has no simply connected neighborhood in U. Therefore CH is not locally simply connected.

Theorem 227. *[Problem 11-20] Suppose X is a connected space that has a contractible universal covering space. For any connected and locally path-connected space Y, a continuous map $f : Y \to X$ is null-homotopic if and only if for each $y \in Y$, the induced homomorphism $f_* : \pi_1(Y, y) \to \pi_1(X, f(y))$ is the trivial map. This result need not hold if the universal covering space is not contractible.*

Proof. We can assume that Y is nonempty. Let $q : E \to X$ be a covering map with E contractible. Suppose that the induced homomorphism $f_* : \pi_1(Y, y) \to \pi_1(X, f(y))$ is trivial for some $y \in Y$. Choose some $e \in E$ such that $q(e) = f(y)$. By Theorem 11.18, there is a lift $\widetilde{f} : Y \to E$ such that $f = q \circ \widetilde{f}$ and $\widetilde{f}(y) = e$. Let $G : E \times I \to E$ be a deformation retraction of E to a point; then $q \circ G \circ (\widetilde{f} \times \mathrm{Id}_I)$ is a homotopy from f to a constant map. The converse follows from Theorem 147. □

Example 228. [Problem 11-21] For which compact, connected surfaces M do there exist continuous maps $f : M \to \mathbb{S}^1$ that are not null-homotopic? Prove your answer correct.

If $M = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2 = (\mathbb{P}^2)^{\# k}$ (with k copies of the projective plane) then we have an induced homomorphism $f_* : \pi_1((\mathbb{P}^2)^{\# k}) \to \mathbb{Z}$. If $k = 1$ then the only such homomorphism is the trivial map, so f must be null-homotopic by Theorem 227. If $M = \mathbb{S}^2$ then all continuous maps $f : M \to \mathbb{S}^1$ are null-homotopic since $\pi_1(\mathbb{S}^2)$ is trivial. If $M = (\mathbb{T}^2)^{\# k}$ then f might not be null-homotopic.

Chapter 12. Group Actions and Covering Maps

Theorem 229. *[Exercise 12.12] For any covering map $q : E \to X$, the action of $\mathrm{Aut}_q(E)$ on E is a covering space action.*

Proof. If $e \in E$ then we can choose a neighborhood U of e such that $q(U)$ is open and $q|_U$ is a homeomorphism. If $\varphi \in \mathrm{Aut}_q(E)$ and $x \in U \cap \varphi(U)$ then $x = \varphi(y)$ for some $y \in U$. By Proposition 12.1 we have that $y \in q^{-1}(\{q(x)\})$, so $y = x$ since $q|_U$ is a homeomorphism. Applying Proposition 12.1 again shows that φ must be the identity map. □

Theorem 230. *[Exercise 12.13] Given a covering space action of a group Γ on a topological space E, the restriction of the action to any subgroup of Γ is a covering space action.*

Proof. Obvious. □

Theorem 231. *[Problem 12-1] Suppose $q_1 : E \to X_1$ and $q_2 : E \to X_2$ are normal coverings. There exists a covering $X_1 \to X_2$ making the obvious diagram commute if and only if $\mathrm{Aut}_{q_1}(E) \subseteq \mathrm{Aut}_{q_2}(E)$.*

Proof. Choose any $e \in E$, let $x_1 = q_1(e)$ and let $x_2 = q_2(e)$. If there is a covering $q : X_1 \to X_2$ such that $q \circ q_1 = q_2$ then for every $\varphi \in \mathrm{Aut}_{q_1}(E)$ we have $q_2 \circ \varphi = q \circ q_1 \circ \varphi = q \circ q_1 = q_2$, i.e. $\varphi \in \mathrm{Aut}_{q_2}(E)$. Conversely, if $\mathrm{Aut}_{q_1}(E) \subseteq \mathrm{Aut}_{q_2}(E)$ then by Theorem 12.14 we have a chain of covering maps

$$E \xrightarrow{\pi_1} E/\operatorname{Aut}_{q_1}(E) \xrightarrow{\widetilde{q_2}} X_2$$

such that $q_2 = \widetilde{q_2} \circ \pi_1$. Since there is a homeomorphism $\psi : X_1 \to E/\operatorname{Aut}_{q_1}(E)$ such that $\psi \circ q_1 = \pi_1$ by Theorem 236, the map $\widetilde{q_2} \circ \psi$ is the desired covering map. □

Example 232. [Problem 12-2] Let $q : X_3 \to X_2$ be the covering map of Example 207.

(1) Determine the automorphism group $\mathrm{Aut}_q(X_3)$.
(2) Determine whether q is a normal covering.
(3) For each of the following maps $f : \mathbb{S}^1 \to X_2$, determine whether f has a lift to X_3 taking 1 to 1.
  (a) $f(z) = z$.
  (b) $f(z) = z^2$.
  (c) $f(z) = 2 - z$.
  (d) $f(z) = 2 - z^2$.

It is easy to check manually that q is a normal covering. Example 11.17 shows that if we write $\pi_1(X_3, 1) = \langle \omega_1, \omega_2, \omega_3 \mid \emptyset \rangle$ where ω_i goes counterclockwise around the ith circle and similarly $\pi_1(X_2, 1) = \langle a, b \mid \emptyset \rangle$ then $q_*\pi_1(X_3, 1) = \langle a, b^2, bab^{-1} \rangle$. Therefore $\mathrm{Aut}_q(X_3) \cong \langle a, b \mid a, b^2, bab^{-1} \rangle$. For (3), we have the following images of $\pi_1(\mathbb{S}^1, 1)$ under f_*: $\langle a \rangle$, $\langle a^2 \rangle$, $\langle b \rangle$, $\langle b^2 \rangle$. So all maps except for (c) have a lift to X_3.

Example 233. [Problem 12-3] Let X_n be the union of n circles described in Problem 10-9, and let A, B, C, and D denote the unit circles centered at 0, 2, 4, and 6, respectively. Define a covering map $q : X_4 \to X_2$ by

$$q(z) = \begin{cases} z, & z \in A, \\ 2 - (2 - z)^2, & z \in B, \\ (z - 4)^2, & z \in C, \\ z - 4, & z \in D. \end{cases}$$

(1) Identify the subgroup $q_*\pi_1(X_4, 1) \subseteq \pi_1(X_2, 1)$ in terms of the generators described in Example 11.17.
(2) Prove that q is not a normal covering map.

Let $\omega_1, \ldots, \omega_4$ be loops that go once counterclockwise around A, B, C and D, starting that 1, 1, 3 and 5. Let c_1 be the lower half of B and let c_2 be the lower half of C. Then $\pi_1(X_4, 1)$ is the free group on $\{[\omega_1], [\omega_2], [c_1 \cdot \omega_3 \cdot \overline{c_1}], [c_1 \cdot c_2 \cdot \omega_4 \cdot \overline{c_2} \cdot \overline{c_1}]\}$, and $q_*\pi_1(X_4, 1) = G = \langle a, b^2, ba^2b^{-1}, bab^{-1}a^{-1}b^{-1} \rangle$. But G is not normal in $\pi_1(X_2, 1)$ since $b \in a^{-1}b^{-1}Gba$ but $b \notin G$, so the covering map q is not normal.

Example 234. [Problem 12-4] Let $\mathcal{E}$ be the figure-eight space of Example 7.32, and let X be the union of the x-axis with infinitely many unit circles centered at $\{2\pi k + i : k \in \mathbb{Z}\}$. Let $q : X \to \mathcal{E}$ be the map that sends each circle in X onto the upper circle in $\mathcal{E}$ by translating in the x-direction and sends the x-axis onto the lower circle by $x \mapsto ie^{ix} - i$. You may accept without proof that q is a covering map.

(1) Identify the subgroup $q_*\pi_1(X, 0)$ of $\pi_1(\mathcal{E}, 0)$ in terms of the generators for $\pi_1(\mathcal{E}, 0)$.
(2) Determine the automorphism group $\mathrm{Aut}_q(X)$.

(3) Determine whether q is a normal covering.

Write $\pi_1(\mathcal{E},0) = \langle a,b \mid \emptyset\rangle$, where a represents a loop that traverses the top circle counterclockwise and b traverses the bottom circle counterclockwise. Since X is homotopic to the wedge sum of infinitely many circles, $\pi_1(X,0)$ is the free group on $\{\omega_k : k \in \mathbb{Z}\}$, where ω_k is a loop that approaches $(2\pi k, 0)$ along the x-axis, traverses the circle centered at $2\pi k + i$ counterclockwise, and returns to $(0,0)$. It is clear that $[q\circ\omega_k] = b^k a b^{-k}$, so $q_*\pi_1(X,0)$ is generated by $\left\{b^k a b^{-k} : k \in \mathbb{Z}\right\}$. This group is normal in $\pi_1(\mathcal{E},0)$, so q is a normal covering and $\operatorname{Aut}_q(X) \cong \mathbb{Z}$.

Theorem 235. *[Problem 12-5] Let $q : E \to X$ be a covering map. The discrete topology is the only topology on $\operatorname{Aut}_q(E)$ for which its action on E is continuous.*

Proof. Fix $x \in E$ and define $F : \operatorname{Aut}_q(E) \to E$ by $F(\varphi) = \varphi(x)$; if the action of $\operatorname{Aut}_q(E)$ on E is continuous then F must be continuous. By Proposition 12.1, F is injective. Let $\varphi \in \operatorname{Aut}_q(E)$, $y = q(\varphi(x)) = q(x)$, let U be an evenly covered neighborhood of y and let $\widetilde{U}$ be the component of $q^{-1}(U)$ containing $\varphi(x)$. Then $F^{-1}(\widetilde{U}) = \{\varphi\}$ is open, which shows that $\operatorname{Aut}_q(E)$ is discrete. □

Theorem 236. *[Problem 12-7] Suppose $q : E \to X$ is a covering map (not necessarily normal). Let $E' = E/\operatorname{Aut}_q(E)$ be the orbit space, and let $\pi : E \to E'$ be the quotient map. Then there is a covering map $q' : E' \to X$ such that $q' \circ \pi = q$.*

Proof. Since q is constant on the fibers of π, it descends to a continuous map $q' : E' \to X$ such that $q' \circ \pi = q$, and it remains to show that q' is a covering map. Let $x \in X$ and let U be an evenly covered neighborhood of x. We have $(q')^{-1}(U) = \pi(q^{-1}(U))$, so it remains to show that $\pi(q^{-1}(U))$ is the disjoint union of open sets that are mapped homeomorphically onto U by q'. But Proposition 12.1 shows that each element of $\operatorname{Aut}_q(E)$ permutes the components of $q^{-1}(U)$, so it is clear that $\pi(q^{-1}(U))$ is the disjoint union of open sets. If U' is one of these open sets and $\widetilde{U}$ is component of $\pi^{-1}(U')$ then $\pi|_{\widetilde{U}}$ and $q|_{\widetilde{U}}$ are homeomorphisms, so $(q')|_{U'} = q|_{\widetilde{U}} \circ (\pi|_{\widetilde{U}})^{-1}$ is also a homeomorphism. □

Example 237. [Problem 12-8] Consider the action of $\mathbb{Z}$ on $\mathbb{R}^m \setminus \{0\}$ defined by $n \cdot x = 2^n x$.

(1) Show that this is a covering space action.
(2) Show that the orbit space $(\mathbb{R}^m \setminus \{0\})/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^{m-1} \times \mathbb{S}^1$.
(3) Show that if $m \geq 2$, the universal covering space of $\mathbb{S}^m \times \mathbb{S}^1$ is homeomorphic to $\mathbb{R}^{m+1} \setminus \{0\}$.

(1) is obvious. Let $q : \mathbb{R}^m \setminus \{0\} \to (\mathbb{R}^m \setminus \{0\})/\mathbb{Z}$ be the quotient map. For (2), define $f : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^{m-1} \times \mathbb{S}^1$ by

$$x \mapsto \left(\frac{x}{\|x\|}, \exp\left(2\pi i \log_2 \|x\|\right)\right).$$

Since f is constant on each orbit, it descends to a continuous map $\widetilde{f} : (\mathbb{R}^m \setminus \{0\})/\mathbb{Z} \to \mathbb{S}^{m-1} \times \mathbb{S}^1$ such that $f = \widetilde{f} \circ q$. Let $\theta : \mathbb{S}^1 \to [0,1]$ be an inverse to $s \mapsto \exp(2\pi i s)$ such that $\theta(1) = 0$ and $\theta|_{\mathbb{S}^1 \setminus \{1\}}$ is continuous. Define $g : \mathbb{S}^{m-1} \times \mathbb{S}^1 \to (\mathbb{R}^m \setminus \{0\})/\mathbb{Z}$ by

$$(s,t) \mapsto q(2^{\theta(t)} s);$$

then g is continuous. Since $f \circ g$ and $g \circ f$ are identity maps, this shows that $(\mathbb{R}^m \setminus \{0\})/\mathbb{Z} \approx \mathbb{S}^{m-1} \times \mathbb{S}^1$. By Theorem 12.14, q is a (normal) covering map. If $m \geq 2$ then $\mathbb{R}^{m+1} \setminus \{0\}$ is simply connected, so the universal covering space of $\mathbb{S}^{m-1} \times \mathbb{S}^1$ is homeomorphic to $\mathbb{R}^{m+1} \setminus \{0\}$.

Example 238. [Problem 12-11] Let $M = \mathbb{T}^2 \# \mathbb{T}^2$.

(1) Show that the fundamental group of M has a subgroup of index 2.
(2) Prove that there exists a manifold $\widetilde{M}$ and a two-sheeted covering map $q : \widetilde{M} \to M$.

We have $\pi_1(M) \cong \mathbb{Z}^2 * \mathbb{Z}^2$ which has the presentation $\langle S \mid R \rangle$, where $S = \{\alpha, \beta, \gamma, \delta\}$ and $R = \{\alpha\beta\alpha^{-1}\beta^{-1}, \gamma\delta\gamma^{-1}\delta^{-1}\}$. But $\mathbb{Z} \cong \langle S \mid R \cup \{\beta, \gamma, \delta\} \rangle$, so $\mathbb{Z}_2 \cong \langle S \mid R \cup \{\alpha^2, \beta, \gamma, \delta\} \rangle$. It follows from Theorem 178 that the index of the normal closure G of $\{\alpha^2, \beta, \gamma, \delta\}$ in $\pi_1(M)$ is 2. By Theorem 12.18, there is a covering map $q : \widetilde{M} \to M$ such that $q_* \pi_1(\widetilde{M}) \cong G$, and by Corollary 12.8, $\operatorname{Aut}_q(\widetilde{M})$ has order 2. This implies that q is two-sheeted.

Example 239. [Problem 12-14] Give an example to show that a subgroup of a finitely generated nonabelian group need not be finitely generated.

See Example 234.

Theorem 240. *[Problem 12-15] Suppose X is a topological space that has a universal covering space. Let $x \in X$, and write $G = \pi_1(X, x)$. Let Cov_X denote the category whose objects are coverings of X and whose morphisms are covering homomorphisms; and let Set_G denote the category whose objects are transitive right G-sets and whose morphisms are G-equivariant maps. Define a functor $\mathcal{F} : \mathsf{Cov}_X \to \mathsf{Set}_G$ as follows: for any covering $q : E \to X$, $\mathcal{F}(q)$ is the set $q^{-1}(\{x\})$ with its monodromy action; and for any covering homomorphism $\varphi : E_1 \to E_2$, $\mathcal{F}(\varphi)$ is the restriction of φ to $q^{-1}(\{x\})$. Then $\mathcal{F}$ is an equivalence of categories.*

Proof. Let $S \in \mathsf{Set}_G$. By Theorem 12.18, there is a covering map $q : E \to X$ such that the conjugacy class of $q_* \pi_1(E)$ is the same as the isotropy type of S, and by Theorem 11.29, the isotropy type of $\mathcal{F}(q)$ is equal to the isotropy type of S. Then $\mathcal{F}(q)$ is isomorphic to S by Proposition 11.26. Furthermore, Proposition 12.1 shows that $\mathcal{F} : \operatorname{Hom}_{\mathsf{Cov}_X}(q_1, q_2) \to \operatorname{Hom}_{\mathsf{Set}_G}(\mathcal{F}(q_1), \mathcal{F}(q_2))$ is injective. Suppose $q_i : E_i \to X$ are covering maps for $i = 1, 2$ and $f : \mathcal{F}(q_1) \to \mathcal{F}(q_2)$ is G-equivariant. Choose

$e_1 \in \mathcal{F}(q_1)$ and let $e_2 = f(e_1) \in \mathcal{F}(q_2)$; then $G_{e_1} \subseteq G_{e_2}$ by Proposition 11.24, and $q_*\pi_1(E_1, e_1) \subseteq q_*\pi_1(E_2, e_2)$ by Theorem 11.29. Applying Theorem 11.37 shows that there is a covering homomorphism $q : E_1 \to E_2$ from q_1 to q_2 taking e_1 to e_2. Since $\mathcal{F}(q)$ agrees with f at e_1, we must have $\mathcal{F}(q) = f$ by Proposition 11.24. □

Theorem 241. *[Problem 12-16] Suppose G is a topological group acting continuously on a Hausdorff space E. If the map $\alpha : G \times E \to E$ defining the action is a proper map, then the action is a proper action.*

Proof. Define $\Theta : G \times E \to E \times E$ by $(g, e) \mapsto (\alpha(g, e), e)$; we want to show that Θ is a proper map. Let $\pi : E \times E \to E$ be the projection onto the first coordinate. Suppose $L \subseteq E \times E$ is compact; then $\pi(L)$ is also compact, and $\alpha^{-1}(\pi(L))$ is compact since α is proper. Since E is Hausdorff, L is closed and $\Theta^{-1}(L)$ is closed. But $\Theta^{-1}(L) \subseteq \alpha^{-1}(\pi(L))$, so $\Theta^{-1}(L)$ must be compact. □

Chapter 13. Homology

Definition 242. Given a continuous map $f : X \to Y$, let $f_\# : C_p(X) \to C_p(Y)$ be the homomorphism defined by setting $f_\#\sigma = f \circ \sigma$ for each singular p-simplex σ. If $c \in Z_p(X)$ then $\partial(f_\# c) = f_\#(\partial c) = 0$, so $f_\# Z_p(X) \subseteq Z_p(Y)$. Similarly, if $c \in B_p(X)$ then $c = \partial b$ for some $b \in C_{p+1}(X)$, so $f_\# c = f_\#(\partial b) = \partial(f_\# b)$ which shows that $f_\# B_p(X) \subseteq B_p(Y)$. Let $\pi_X : Z_p(X) \to H_p(X)$ and $\pi_Y : Z_p(Y) \to H_p(Y)$ be the quotient maps. Since $\pi_Y \circ f_\# : Z_p(X) \to H_p(Y)$ satisfies $B_p(X) \subseteq \ker(\pi_Y \circ f_\#)$, there is a unique homomorphism $f_* : H_p(X) \to H_p(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} Z_p(X) & \xrightarrow{f_\#} & Z_p(Y) \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ H_p(X) & \xrightarrow[f_*]{} & H_p(Y) \end{array}$$

This map is called the **homomorphism induced by** f.

Theorem 243. *Let X, Y, and Z be topological spaces.*

(1) *The homomorphism $(\mathrm{Id}_X)_* : H_p(X) \to H_p(X)$ induced by the identity map of X is the identity of $H_p(X)$.*
(2) *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then*

$$(g \circ f)_* = g_* \circ f_* : H_p(X) \to H_p(Z).$$

Thus the pth singular homology group defines a covariant functor from the category of topological spaces to the category of abelian groups.

Proof. Part (1) follows from the fact that $(\mathrm{Id}_X)_\# = \mathrm{Id}_{C_p(X)}$ and that $\mathrm{Id}_{H_p(X)}$ satisfies the diagram in Definition 242. For (2), since $(g \circ f)_\# = g_\# \circ f_\#$ we have

$$g_* \circ f_* \circ \pi_X = g_* \circ \pi_Y \circ f_\# = \pi_Z \circ g_\# \circ f_\# = \pi_Z \circ (g \circ f)_\#.$$

Therefore $(g \circ f)_\# = g_* \circ f_*$ by uniqueness. □

Theorem 244. *[Exercise 13.10] Suppose $f : X \to Y$ is a homotopy equivalence. Then for each $p \geq 0$, $f_* : H_p(X) \to H_p(Y)$ is an isomorphism.*

Proof. Let $g : Y \to X$ be a continuous map such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. By Theorem 13.8 we have

$$g_* \circ f_* = (g \circ f)_* = \mathrm{Id}_{H_p(X)} \quad \text{and} \quad f_* \circ g_* = (f \circ g)_* = \mathrm{Id}_{H_p(Y)},$$

so f_* is an isomorphism. □

Theorem 245. *[Exercise 13.12] If $F, G : C_* \to D_*$ are chain homotopic maps, then $F_* = G_* : H_p(C_*) \to H_p(D_*)$ for all p.*

Proof. Let $h : C_p \to D_{p+1}$ be a chain homotopy from F to G. Fix some p and let $\pi_C, \pi_D : \ker \partial_p \to \ker \partial_p / \operatorname{im} \partial_{p+1}$ be the quotient maps. For all $c \in \ker \partial_p$ we have

$$\begin{aligned}(G_* \circ \pi_C)(c) &= (\pi_D \circ G)(c)\\ &= \pi_D(F(c) + (G - F)(c))\\ &= \pi_D(F(c) + (h \circ \partial + \partial \circ h)(c))\\ &= \pi_D(F(c) + (\partial \circ h)(c))\\ &= (\pi_D \circ F)(c)\end{aligned}$$

since $(\partial \circ h)(c) \in \operatorname{im} \partial_{p+1}$. By uniqueness, $F_* = G_*$. □

Theorem 246. *Let*

$$0 \to C_* \xrightarrow{F} D_* \xrightarrow{G} E_* \to 0$$

be a short exact sequence of chain maps. Then for each p there is a connecting homomorphism $\partial_* : H_p(E_*) \to H_{p-1}(C_*)$ such that the following sequence is exact:

$$(*) \qquad \cdots \xrightarrow{\partial_*} H_p(C_*) \xrightarrow{F_*} H_p(D_*) \xrightarrow{G_*} H_p(E_*) \xrightarrow{\partial_*} H_{p-1}(C_*) \xrightarrow{F_*} \cdots.$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{p+1} & \xrightarrow{F} & D_{p+1} & \xrightarrow{G} & E_{p+1} & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_p & \xrightarrow{F} & D_p & \xrightarrow{G} & E_p & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_{p-1} & \xrightarrow{F} & D_{p-1} & \xrightarrow{G} & E_{p-1} & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_{p-2} & \xrightarrow{F} & D_{p-2} & \xrightarrow{G} & E_{p-2} & \longrightarrow & 0.
\end{array}$$

Every square in this diagram commutes, and the horizontal rows are exact. Suppose $e \in E_p$ with $\partial_p e = 0$. Since G is surjective, there is some $d \in D_p$ such that $e = Gd$. Then $G\partial_p d = \partial_p Gd = \partial_p e = 0$, so $\partial_p d \in \ker G$ and $\partial_p d = Fc$ for some $c \in C_{p-1}$ by exactness at D_{p-1}. We have $F\partial_{p-1}c = \partial_{p-1}Fc = \partial_{p-1}\partial_p d = 0$, so $\partial_{p-1}c = 0$ since F is injective. Therefore we can define a map $\partial_\# : \ker \partial_p \to H_{p-1}(C_*)$ by setting $\partial_\# e = \pi_C c$ where $\pi_C : \ker \partial_p \to H_{p-1}(C_*)$ is the quotient map. We first ensure that $\partial_\#$ is well-defined. Suppose we have $d' \in D_p$ and $c' \in C_{p-1}$ with $e = Gd'$ and $\partial_p d' = Fc'$. Then $d - d' \in \ker G$, so $d - d' = Fa$ for some $a \in C_p$. Now $F(c - c') = \partial_p(d - d') = \partial_p Fa = F\partial_p a$, so $c - c' = \partial_p a$ since F is injective. We have $\pi_C c = \pi_C c'$, which shows that $\partial_\#$ is well-defined. Next, we show that $\partial_\#$ is a homomorphism. If $e' \in E_p$ then $e' = Gd'$ and $\partial_p d' = Fc'$ for some $d' \in D_p$ and $c' \in C_{p-1}$. So $e+e' = G(d+d')$ and $\partial_p(d+d') = F(c+c')$, which implies that $\partial_\#(e + e') = \partial_\# e + \partial_\# e'$. Finally, we show that $\operatorname{im} \partial_{p+1} \subseteq \ker \partial_\#$. Suppose that $e = \partial_{p+1}e'$ for some $e' \in E_{p+1}$; then $e' = Gd'$ for some $d' \in D_{p+1}$. We have $G\partial_{p+1}d' = \partial_{p+1}Gd' = \partial_{p+1}e' = e = Gd$, so $d - \partial_{p+1}d' \in \ker G$ and $d - \partial_{p+1}d' = Fa$ for some $a \in C_p$. Now $Fc = \partial_p d = \partial_p(d - \partial_{p+1}d' + \partial_{p+1}d') = \partial_p Fa = F\partial_p a$, so $c = \partial_p a$ since F is injective. Then $\pi_C c = 0$, which shows that $\operatorname{im} \partial_{p+1} \subseteq \ker \partial_\#$. Therefore $\partial_\#$ descends to a homomorphism $\partial_* : H_p(E_*) \to H_{p-1}(C_*)$ such that $\partial_* \pi_E = \partial_\#$, where $\pi_E : \ker \partial_p \to H_p(E_*)$ is the quotient map.

Now we prove exactness of the sequence (*). Given a cycle c, we write $[c]$ for the homology class of c. Suppose $[c] \in H_p(C_*)$ with $[c] = \partial_*[e]$ for some $[e] \in H_p(E_*)$. From the definition of ∂_* there is some $d \in D_{p+1}$ such that $\partial_{p+1}d = Fc$, so $F_*[c] = [Fc] = 0$. Therefore $\operatorname{im} \partial_* \subseteq \ker F_*$. Conversely, if $F_*[c] = [Fc] = 0$ then $Fc = \partial_{p+1}d$ for some $d \in D_{p+1}$. We have $\partial_{p+1}Gd = G\partial_{p+1}d = GFc = 0$, and $\partial_*[Gd] = [c]$ from the definition of ∂_*. This shows that $\ker F_* \subseteq \operatorname{im} \partial_*$, and proves exactness at $H_p(C_*)$. Next, we prove exactness at $H_p(D_*)$. Since $GF = 0$ and $G_*F_* = 0$, it is immediate that $\operatorname{im} F_* \subseteq \ker G_*$. Suppose $G_*[d] = 0$, i.e. $Gd = \partial_{p+1}e$ for some $e \in E_{p+1}$. Since G is surjective, there is some $d' \in D_{p+1}$ such that $e = Gd'$. Then $G\partial_{p+1}d' = \partial_{p+1}Gd' = \partial_{p+1}e = Gd$, so $d - \partial_{p+1}d' \in \ker G$ and $d - \partial_{p+1}d' = Fc$ for some $c \in C_p$. Since $F_*[c] = [Fc] = [d]$, this shows that $\ker G_* \subseteq \operatorname{im} F_*$ and proves exactness at $H_p(D_*)$. Finally, we prove exactness at $H_p(E_*)$. Suppose $[e] \in H_p(E_*)$ such that $[e] = G_*[d] = [Gd]$ for some $d \in D_p$ with $\partial_p d = 0$. Let $[c] = \partial_*[e] = \partial_*[Gd]$. From the definition of ∂_* we have $Fc = \partial_p d = 0$, so $c = 0$ since F is injective. This shows that $\operatorname{im} G_* \subseteq \ker \partial_*$. Conversely, suppose $[e] \in H_p(E_*)$ with $\partial_*[e] = 0$. This means that $e = Gd$ and $\partial_p d = Fc$ for some $d \in D_p$ and some boundary $c \in C_{p-1}$. Choose $c' \in C_p$ such that $c = \partial_p c'$. Then $\partial_p d = F\partial_p c' = \partial_p Fc'$, so $\partial_p(d - Fc') = 0$ and $G(d - Fc') = Gd = e$. Therefore $G_*[d - Fc'] = [Gd] = [e]$, so $\ker \partial_* \subseteq \operatorname{im} G_*$. This proves exactness at $H_p(E_*)$. □

Theorem 247. *[Exercise 13.39] The induced cohomology homomorphism satisfies the following properties.*

1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)^* = f^* \circ g^*$.*
2. *The homomorphism induced by the identity map is the identity.*

Therefore, the assignments $X \mapsto H^p(X; G)$, $f \mapsto f^*$ define a contravariant functor from the category of topological spaces to the category of abelian groups. Furthermore, if $f : X \to Y$ is a homeomorphism then for every abelian group G and every integer $p \geq 0$ the map $f^* : H^p(Y; G) \to H^p(X; G)$ is an isomorphism.

Proof. We have

$$((g \circ f)^\# \varphi)(c) = \varphi((g \circ f)_\# c) = \varphi((g_\# \circ f_\#)(c)) = (g^\# \varphi)(f_\# c) = ((f^\# \circ g^\#)\varphi)(c),$$

so $(g \circ f)^\# = f^\# \circ g^\#$ and (1) follows. For (2) we have $((\mathrm{Id}_X)^\# \varphi)(c) = \varphi((\mathrm{Id}_X)_\# c) = \varphi(c)$, so $(\mathrm{Id}_X)^\# = \mathrm{Id}_{C^p(X;G)}$. □

Theorem 248. *[Problem 13-1] Let $X_1, \dots, X_k$ be spaces with nondegenerate base points. Then for every $p > 0$,*

$$H_p(X_1 \vee \cdots \vee X_k) \cong H_p(X_1) \oplus \cdots \oplus H_p(X_k).$$

Proof. By Lemma 10.6, it suffices to prove the theorem for the case $k = 2$. Let p_1, p_2 be nondegenerate base points. For $i = 1, 2$, let W_i be a neighborhood of p_i that admits a strong deformation retraction onto $\{p_i\}$. Let $q : X_1 \amalg X_2 \to X_1 \vee X_2$ be the quotient map, let $U = q(X_1 \amalg W_2)$ and let $V = q(W_1 \amalg X_2)$. Since $X_1 \amalg W_2$ and $W_1 \amalg X_2$ are saturated open sets in $X_1 \amalg X_2$, the sets U and V are open in $X_1 \vee X_2$. Applying Theorem 13.16 and noting that $U \cap V$ is contractible, we have an exact sequence

$$0 \to H_p(U) \oplus H_p(V) \xrightarrow{k_* - l_*} H_p(X_1 \vee X_2) \to 0$$

where $k : U \to X_1 \vee X_2$ and $l : V \to X_1 \vee X_2$ are the inclusions. This implies that $k_* - l_*$ is an isomorphism. Since X_1 is a deformation retract of U and X_2 is a deformation retract of V, we have $H_p(X_1) \oplus H_p(X_2) \cong H_p(U) \oplus H_p(V) \cong H_p(X_1 \vee X_2)$ as desired. □

Theorem 249. *[Problem 13-2] (cf. Theorem 157)*

(1) *If $U \subseteq \mathbb{R}^n$ is an open subset with $n \geq 2$ and $x \in U$, then $H_{n-1}(U \setminus \{x\}) \neq 0$.*
(2) *If $m > n$ then $\mathbb{R}^m$ is not homeomorphic to any open subset of $\mathbb{R}^n$.*

Proof. For (1), we can assume $n > 2$ for the case $n = 2$ is covered by Theorem 157. Let $B_r(x) \subseteq U$ be an open ball around x. Applying Theorem 13.16 with the open subsets $B_r(x) \setminus \{x\}$ and $U \setminus \overline{B_{r/2}(x)}$ gives an exact sequence

$$H_{n-1}(B_r(x) \setminus \{x\}) \oplus H_{n-1}(U \setminus \overline{B_{r/2}(x)}) \to H_{n-1}(U \setminus \{x\}) \to H_{n-2}(B_r(x) \setminus \overline{B_{r/2}(x)}).$$

Since $B_r(x) \setminus \overline{B_{r/2}(x)}$ is homotopic to $\mathbb{S}^{n-1}$ and $B_r(x) \setminus \{x\} \cong \mathbb{R}^n \setminus \{0\}$, the sequence reduces to

$$\mathbb{Z} \oplus H_{n-1}(U \setminus \overline{B_{r/2}(x)}) \to H_{n-1}(U \setminus \{x\}) \to 0.$$

This shows that $H_{n-1}(U \setminus \{x\}) \neq 0$. For (2), the cases $n = 1, 2$ are covered by Theorem 80 and Theorem 157. Suppose $U \subseteq \mathbb{R}^n$ is open with $n > 2$ and $\varphi : U \to \mathbb{R}^m$ is a homeomorphism. Choose any $x \in U$; then $U \setminus \{x\} \approx \mathbb{R}^m \setminus \{\varphi(x)\}$, so $H_{n-1}(U \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^m \setminus \{\varphi(x)\}) = 0$. This contradicts part (1). □

Theorem 250. *[Problem 13-3] A nonempty topological space cannot be both an m-manifold and an n-manifold for any $m > n$ (cf. Theorem 158).*

Proof. The cases $n = 1, 2$ are covered by Theorem 81 and Theorem 158, so we can assume that $n > 2$. Let M be a nonempty topological space that is both a m-manifold and an n-manifold for $m > n > 2$. Choose some $p \in M$ and let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homeomorphisms where U_1 and U_2 are neighborhoods of p, V_1 is open in $\mathbb{R}^m$, and V_2 is open in $\mathbb{R}^n$. Let B be an open ball around $\varphi_1(p)$ contained in $\varphi_1(U_1 \cap U_2)$. Then $B \approx (\varphi_2 \circ \varphi_1^{-1})(B)$, but this contradicts Theorem 249. □

Theorem 251. *[Problem 13-4] Suppose M is an n-dimensional manifold with boundary. Then the interior and boundary of M are disjoint (cf. Theorem 159).*

Proof. Suppose $p \in M$ is both an interior and boundary point. Choose coordinate charts (U, φ) and (V, ψ) such that U, V are neighborhoods of p, $\varphi(U)$ is open in $\operatorname{Int} \mathbb{H}^n$, $\psi(V)$ is open in $\mathbb{H}^n$, and $\psi(p) \in \partial \mathbb{H}^n$. Let $W = U \cap V$; then $\varphi(W)$ is homeomorphic to $\psi(W)$. But this is impossible, for $H_{n-1}(\varphi(W) \setminus \{\varphi(p)\}) \neq 0$ by Theorem 249 while $H_{n-1}(\psi(W) \setminus \{\psi(p)\}) = 0$. □

Theorem 252. *[Problem 13-5] Let $n \geq 1$. If $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map that has a continuous extension to a map $F : \overline{\mathbb{B}^{n+1}} \to \mathbb{S}^n$, then f has degree zero (cf. Theorem 160).*

Proof. Define a homotopy $H : \mathbb{S}^n \times I \to \mathbb{S}^n$ from a constant map to f by $H(x, t) = F(tx)$; then $\deg f = 0$ by Proposition 13.25 and Proposition 13.27. □

Theorem 253. *[Problem 13-6] $\mathbb{S}^n$ is not a retract of $\overline{\mathbb{B}^{n+1}}$ for any n.*

Proof. The case $n = 0$ is clear, so we can assume $n \geq 1$. Since $\overline{\mathbb{B}^{n+1}}$ is contractible, $H_n(\overline{\mathbb{B}^{n+1}}) = 0$. But $H_n(\mathbb{S}^n) = \mathbb{Z}$, so $\mathbb{S}^n$ cannot be a retract of $\overline{\mathbb{B}^{n+1}}$ by Corollary 13.4. □

Theorem 254. *[Problem 13-7] For every integer $n \geq 0$, every continuous map $f : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ has a fixed point (cf. Theorem 162).*

Proof. We can assume that $n \geq 1$. If f has no fixed point then we can define a continuous map $\varphi : \overline{\mathbb{B}^n} \to \mathbb{S}^{n-1}$ by

$$\varphi(x) = \frac{x - f(x)}{\|x - f(x)\|}.$$

This contradicts Theorem 252, since we can define a homotopy $H : \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$ from $\varphi|_{\mathbb{S}^{n-1}}$ to $\mathrm{Id}_{\mathbb{S}^{n-1}}$ by

$$H(x,t) = \frac{x - (1-t)f(x)}{\|x - (1-t)f(x)\|}$$

using an argument identical to Theorem 162. □

Theorem 255. *[Problem 13-8] If n is even then $\mathbb{Z}_2$ is the only nontrivial group that can act freely on $\mathbb{S}^n$ by homeomorphisms.*

Proof. Suppose G acts freely on $\mathbb{S}^n$ by homeomorphisms. For any $g \in G$, write φ_g for the homeomorphism $x \mapsto g \cdot x$. Then $\deg \varphi_g = \pm 1$ by Proposition 13.25, so deg defines a homomorphism from G to $\{\pm 1\}$. If $\deg \varphi_g = 1$ and $\varphi_g \neq \mathrm{Id}_{\mathbb{S}^n}$ then φ_g has no fixed point since G acts freely. But $\deg \varphi = -1$ by Theorem 13.29, which is a contradiction. This shows that deg is an injective homomorphism, so G is either the trivial group or is isomorphic to $\mathbb{Z}_2$. □

Example 256. [Problem 13-9] Use the CW decomposition of Theorem 117 and the results of this chapter to compute the singular homology groups of the 3-dimensional real projective space $\mathbb{P}^3$.

We first compute $H_2(\mathbb{P}^2)$. By Proposition 13.33 there is a short exact sequence

$$0 \to H_2(\mathbb{P}^1) \to H_2(\mathbb{P}^2) \to K \to 0,$$

where K is the kernel of $((q \circ F)|_{\partial \mathbb{B}^2})_* : H_1(\partial \mathbb{B}^2) \to H_1(\mathbb{P}^1)$. But $H_2(\mathbb{P}^1) = K = 0$, so $H_2(\mathbb{P}^2) = 0$.

Let $\varphi = (q \circ F)|_{\partial \mathbb{B}^3}$ as defined in Theorem 117, and consider $\mathbb{P}^2$ as a subspace of $\mathbb{P}^3$. Since $H_2(\mathbb{P}^2) = 0$, the kernel K of $\varphi_* : H_2(\partial \mathbb{B}^3) \to H_2(\mathbb{P}^2)$ is $H_2(\partial \mathbb{B}^3) = H_2(\mathbb{S}^2) = \mathbb{Z}$ and the image L is 0. It is clear that $H_0(\mathbb{P}^3) = \mathbb{Z}$. By Proposition 13.33, if $p = 1$ or $p > 3$ then the inclusion $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$ induces an isomorphism. Therefore $H_1(\mathbb{P}^3) = H_1(\mathbb{P}^2) = \mathbb{Z}_2$ and $H_p(\mathbb{P}^3) = 0$ for $p > 3$. If $p = 2$ then we have a short exact sequence

$$0 \to L \hookrightarrow H_2(\mathbb{P}^2) \to H_2(\mathbb{P}^3) \to 0.$$

Since $H_2(\mathbb{P}^2) = L = 0$, we have $H_2(\mathbb{P}^3) = 0$. If $p = 3$, we have a short exact sequence

$$0 \to H_3(\mathbb{P}^2) \to H_3(\mathbb{P}^3) \to K \to 0$$

that reduces to an exact sequence

$$0 \to H_3(\mathbb{P}^3) \to \mathbb{Z} \to 0,$$

so $H_3(\mathbb{P}^3) = \mathbb{Z}$. Therefore

$$H_p(\mathbb{P}^3) = \begin{cases} \mathbb{Z}, & p = 0, \\ \mathbb{Z}_2, & p = 1, \\ 0, & p = 2, \\ \mathbb{Z}, & p = 3, \\ 0, & p > 3. \end{cases}$$

Theorem 257. *[Problem 13-11] For any field $\mathbb{F}$ of characteristic zero, the functor $G \mapsto \operatorname{Hom}(G, \mathbb{F})$, $f \mapsto f^*$ is exact.*

Proof. Suppose that the sequence

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

is exact at B; we want to show that

$$\cdots \leftarrow \operatorname{Hom}(A, \mathbb{F}) \xleftarrow{f^*} \operatorname{Hom}(B, \mathbb{F}) \xleftarrow{g^*} \operatorname{Hom}(C, \mathbb{F}) \leftarrow \cdots$$

is exact at $\operatorname{Hom}(B, \mathbb{F})$. If $\beta = g^*\gamma$ for some $\gamma \in \operatorname{Hom}(C, \mathbb{F})$ then for all $a \in A$ we have

$$(f^*g^*\gamma)(a) = (\gamma g f)(a) = 0$$

since $\operatorname{im} f \subseteq \ker g$. Therefore $f^*\beta = f^*g^*\gamma = 0$, which shows that $\operatorname{im} g^* \subseteq \ker f^*$. Conversely, suppose that $f^*\beta = 0$; then $\beta f a = 0$ for all $a \in A$. We define a map $\gamma \in \operatorname{Hom}(\operatorname{im} g, \mathbb{F})$ by setting $\gamma(c) = \beta b$ for any $c = gb \in \operatorname{im} g$. To show that γ is well-defined, suppose $b, b' \in B$ with $gb = gb'$. Then $b - b' \in \ker g$, so $b - b' = fa$ for some $a \in A$, and $\beta b - \beta b' = \beta(b - b') = \beta f a = 0$. It is clear that γ is a homomorphism since g and β are homomorphisms. By Lemma 13.42 there is an extension $\gamma' \in \operatorname{Hom}(C, \mathbb{F})$ of γ, so $\beta = g^*\gamma'$ and $\ker f^* \subseteq \operatorname{im} g^*$ as desired. ∎

Theorem 258. *[Problem 13-12] Let X be a topological space and let $U, V \subseteq X$ be open subsets whose union is X. Then there is an exact Mayer-Vietoris sequence for cohomology with coefficients in a field $\mathbb{F}$ of characteristic zero:*

$$\cdots \to H^{p-1}(U \cap V; \mathbb{F}) \to H^p(X; \mathbb{F}) \to H^p(U; \mathbb{F}) \oplus H^p(V; \mathbb{F}) \to H^p(U \cap V; \mathbb{F}) \to \cdots.$$

Proof. This follows from Theorem 257. ∎

References

[1] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2nd edition, 2013.